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Holliday

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Holliday, Francis

Syntagma Matheſios:

Containing the

RESOLUTION

OF

EQUATIONS:

WITH

A New Way of Solving CUBIC and BIQUADRATIC EQUATIONS, *Analytically* and *Geometrically.*

ALSO

The UNIVERSAL METHOD

OF

Converging SERIES,

After an Eaſy and *Expeditious Manner.*

Wherein alſo are treated

The Series for *Trigonometrical* Operations; ſome new uſeful Properties of *Conics*; Centre of *Oſcillation*; the direct and inverſe Method of the Laws of *Centripetal Forces*; a Variety of *Exponential Equations*; with the Inveſtigation of ſeveral other abſtruſe Problems, *&c.*

To all which is prefixed,

An ESSAY *on the* MATHEMATICS.

Purgationi rationalis Animae diſquiſitionis ſunt Mathematicae.

LONDON:

Printed for J. FULLER, at the *Bible* and *Dove* in *Ave-Mary-Lane*

M DCC XLV.

THE

PREFACE.

I *Imagine it may ſeem a raſh Attempt to write on the Subject of the* Mathematics, *ſeeing the Bookſellers Shelves are already filled with them; but conſidering thoſe Books already publiſhed, there are none that ſhews how to manage an* Equation *higher than a* Quadratic, *or at leaſt in a very tedious, clumſy, round-about Way. Upon theſe Conſiderations I was inclined to believe, that could a ſmall Tract be compiled, ſo as to ſhew how to manage* Equations *in a Method plainer than*

 uſual,

usual, it would meet with a favourable Reception.

Mr. Ralphson *some Years ago published a Book on this Subject, called* Resolutio Equationum; *but as the ingenious Author wrote it in* Latin, *and the* Analytic *Art being vastly improved since its Publication, it is but of very little Use now, especially as he confined himself chiefly to* Quadratics.

I might quote many great Men, did I not chuse rather to draw a Vail over their Defects, from whom we daily reap Advantage in other Things, than to triumph over them on this Account.

My Inclinations therefore led me to pursue this Part, which I hope I have in some Measure answered: Nay, if I should go no farther than to bring down some of the best and most useful Things, already known, to be understood by those of an ordinary Capacity, I should think this would exempt me from Censure.

First, *I have given the Reader some choice Examples for ordering* Equations,

Equations, *with the Rules from Sir* Iſaac Newton's *own Words, beginning in as plain, familiar, and eaſy Method as poſſible, leading him on Step by Step, which I ſhould think would encourage all thoſe that are willing to venture and proceed with Chearfulneſs. And, when the Reader has made himſelf Maſter of Managing* Equations, *where the Exterminating of* Quantities, *and Doctrine of* Surds, *are concerned, then let him proceed to our* Univerſal Method of Converging Series, *wherein are laid down a great Variety of Examples in the plaineſt and eaſieſt Method imaginable.*

Secondly: *At the Deſire of ſome Friends, to render the Work more generally uſeful to thoſe that are acquainted with the more ſublime Branches of the* Mathematics, *I have added ſome curious Pieces tranſlated from the* Latin *of the* Philoſophical Tranſactions; *as ſome uſeful Properties of* Conics; *Sir* Iſaac Newton's Differentials, *or Method of* Fluxions *explained*; Centre

of

of Oscillation, Percussion, &c. *the direct and inverse Method of the Laws of* Centripetal Forces. *And, if I have not rendered the English in such a flowing, elegant Strain, as some curious and ingenious Persons might wish for; or if my Manner of explaining some of those great Truths, and a few of the Consequences I have drawn, should be defective; and perhaps, by some Links being dropt, and from Faults in the Wording, the Chain of Reasoning may not always be clear and strong; yet I am sure the Foundation is solid and just; and perhaps the Method, when managed by a clearer Head, and more solid Judgment, may become a noble Source in divine Knowledge and sublime Philosophy.*

Lastly, *I have added some choice* Problems, *that were proposed in the public Papers by several ingenious Mathematicians, that I might make the Book the more answerable to the Title: And to the Whole is prefixed, An* Essay *on the* Mathematics.

As

As to the Work in general, I think there is nothing left undemonſtrated, that is capable of it; and, if any ſmall Miſtake may have happened, which in a Work of this Nature is not altogether unlikely, I am ſure the generous and honeſt Part of Mankind will excuſe it; and as for thoſe Perſons whoſe Excellency lies in finding Fault with every Thing, and very often when there is no real Occaſion, and of diſcovering Errors where there are none; all I can ſay to them is, that they would produce ſomething better.

F. Holliday.

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[Place this before the Essay.]

I Would have the Reader understand, that the following are Translations from the *Latin* of the *Philosophical Transactions*, *Motte's Abridgment*, Vol I.

And also, I would have the Reader understand, that Part of the *Essay on the Usefulness of Mathematical Learning* was printed in 1701, being a Letter from a Gentleman in Town to his Friend at *Oxford*: But, as it has been long out of Print, I thought it would not be amiss to revive it; which I have done, by carrying it on much further.

An

AN
ESSAY
ON THE
USEFULNESS
OF
Mathematical Learning.

IN all Ages and Countries, where Learning hath prevailed, the *Mathematical Sciences* have been look'd upon as the most considerable Branch of it. The very Name Μάθησις implies no less, by which they were called either for their Excellency, or because, of all the Sciences, they were first taught, or because they were judged to comprehend πάντα τὰ Μαθήματα. And amongst those that are commonly reckoned to be the *Seven Liberal Arts*, four are *Mathematical*, to wit, *Arithmetic*, *Music*, *Geometry* and *Astronomy*.

But notwithstanding their Excellency and Reputation, they have not been taught nor study'd so universally, as some of the rest, which I take to have proceeded from the following Causes:

The Aversion of the greatest part of Mankind to serious Attention and close Arguing; their not comprehending sufficiently the Necessity or great Usefulness of these in other parts of Learning: An Opinion that this Study requires a particular Genius and turn of Head, which few are so happy as to be born with: And the want of publick Encouragement and able Masters. For these, and perhaps some other Reasons, this Study hath been generally neglected, and regarded only by some few Persons, whose happy *Genius* and Curiosity have prompted them to it, or who have been forc'd upon it by its immediate subserviency to some particular Art or Office.

Therefore I think I cannot do better Service to Learning, Youth, and the Nation in general, than by shewing, *that the Mathematics, of all parts of human Knowledge, for the Improvement of the Mind, for their subserviency to other Arts, and their usefulness to the Commonwealth, deserve most to be encouraged.*

I know a Discourse of this Nature will be offensive to some, who, while they are ignorant of *Mathematics*, yet think themselves Masters of all valuable Learning: But their Displeasure must not deter me from delivering an useful Truth.

The Advantages which accrue to the Mind by *Mathematical* Studies, consist chiefly in these Things;

First, In accustoming it to *Attention.* Secondly, In giving it a Habit of *close* and *demonstrative Reasoning.* Thirdly, in freeing it from *Prejudice, Credulity* and *Superstition.*

First, the *Mathematics* make the Mind attentive to the Objects which it considers. This they do by entertaining it with a great variety of Truths,

Truths, which are delightful and evident, but not obvious. Truth is the ſame thing to the Underſtanding, as Muſick to the Ear, and Beauty to the Eye. The purſuit of it does really as much gratify a natural Faculty, implanted in us by our wiſe Creator, as the pleaſing of our Senſes; only in the former Caſe, as the Object and Faculty are more ſpiritual, the Delight is the more pure, free from the Regret, Turpitude, Laſſitude and Intemperance, that commonly attend ſenſual Pleaſures. The moſt part of other Sciences conſiſting only of probable Reaſonings, the Mind has not where to fix, and wanting ſufficient Principles to purſue its Searches upon, gives them over as impoſſible. Again, as in *Mathematical Inveſtigations* Truth may be found, ſo it is not always *obvious*; this ſpurs the Mind, and makes it diligent and attentive. *In Geometriâ* (ſays *Quinctilian*, Lib. 1. cap. 10.) *partem fatentur eſſe utilem teneris Ætatibus: Agitari namque Animos atque acui ingenia, et celeritatem percipiendi venire inde concedunt.* And *Plato* obſerves, that the Youth, who are furniſhed with *Mathematical* Knowledge, are prompt and quick at all other Sciences, εἰς πάντα τὰ Μαθήματα ὀξεῖς φαίνονται. Therefore he calls it κατὰ παιδείαν ὁδὸν. And indeed Youth are generally ſo much more delighted with *Mathematical* Studies, than with the unpleaſant Taſks that are ſometimes impoſed upon them, that I have known ſome reclaimed by them from Idleneſs and neglect of Learning, and acquire in time a Habit of Thinking, Diligence and Attention; Qualities which we ought to ſtudy by all means to beget in their deſultory and roving Minds.

The ſecond Advantage, which the Mind reaps from *Mathematical* Knowledge is, a Habit of *clear*, *demonstrative*, and *methodical* Reasoning. We are contriv'd by Nature, to learn by Imitation more than by Precept: And I believe in that respect, Reasoning is much like other inferior Arts (as Dancing, Singing, *&c.*) acquired by Practice. By accustoming ourselves to reason closely about Quantity, we acquire a Habit of doing so in other things. It is surprizing to see, what superficial, inconsequential Reasonings satisfy the most part of Mankind. A piece of Wit, a Jest, a Simile, or a Quotation of an Author, passes for a mighty Argument: With such things as these are the most part of Authors stuff'd; and from these weighty Premises, they infer their Conclusions. This Weakness and Effeminacy of Mankind in being perswaded, where they are delighted, have made them the Sport of Orators, Poets, and Men of Wit. Those *Lumina Orationis* are indeed very good Diversion for the Fancy, but are not the proper Business of the Understanding; and where a Man pretends to write on Abstract Subjects in a scientifical Method, he ought not to debauch in them. Logical Precepts are more useful, nay they are absolutely necessary for a Rule of formal arguing, in publick Disputations, and confounding an obstinate and perverse Adversary, and exposing him to the Audience, or Readers. *Geometers* will carry a Man further, than all the *Dialectical* Rules. Their *Analysis* is the proper Model we ought to form ourselves upon, and imitate in the regular Disposition and gradual Progress of our Enquiries; and even he who is ignorant of the Nature of *Mathematical Analysis*, uses a Method somewhat Analogous to it. The *Composition* of the *Geometers*, or their Method of demonstrating Truths already found out,

out, *viz.* *By Definitions of Words agreed upon, by Self-evident Truths, and Propositions that have been already demonstrated*, is practicable in other Subjects, though not to the same Perfection, the natural want of Evidence in the things themselves not allowing it; but it is imitable to a considerable Degree. I dare appeal to some Writings of our own Age and Nation, the Authors of which have been *Mathematically* inclined. I shall add no more on this Head, but that one who is accustomed to the methodical Systems of Truths, which the *Geometers* have reared up in the several Branches of those *Sciences* which they have cultivated, will hardly bear with the Confusion and Disorder of other *Sciences* but endeavour as far as he can to reform them.

Thirdly, *Mathematical* Knowledge adds a manly Vigour to the Mind, frees it from *Prejudice*, *Credulity*, and *Superstition*. This it does two ways. First, by accustoming us to examine, and not to take Things upon trust. Secondly, by giving us a clear and extensive Knowledge of the System of the World; which as it creates in us the most profound Reverence of the ALMIGHTY, and Wise *Creator*, so it frees us from the mean and narrow Thoughts, which Ignorance and Superstition are apt to beget. How great an Enemy *Mathematics* are to Superstition, appears from this, that in those Countries where *Romish Priests* exercise their barbarous Tyranny over the Minds of Men, *Astronomers*, who are fully perswaded of the Motion of the Earth, dare not speak out. But tho' the *Inquisition* may extort a Recantation, the *Pope*, and a general Council too, will not find themselves able to perswade to the contrary Opinion. Perhaps this may have given oc-

casion

casion to a Calumnious Suggestion, as if *Mathematics* were an Enemy to Religion, which is a Scandal thrown both on the one and the other; for Truth can never be an Enemy to true Religion, which appears always to the best Advantage, when it is most examined.

——*Si propiùs stes,*
Te capiet magis.——

On the contrary, the Mathematics are Friends to Religion, inasmuch as they charm the Passions, restrain the Impetuosity of Imagination, and purge the Mind from Error and Prejudice.

Vice is Error, Confusion and false Reasoning, and all Truth is more or less opposite to it. Besides, *Mathematical* Studies may serve for a pleasant Entertainment for those Hours, which young Men are apt to throw away upon their Vices; the delightfulness of them being such, as to make Solitude not only easy, but desirable.

What I have said may serve to recommend *Mathematics* for acquiring a vigorous Constitution of Mind, for which purpose they are as useful, as exercise is for procuring Health and Strength to the Body. I proceed now to shew their vast Extent and Usefulness in other Parts of Knowledge. And here it might suffice to tell you, that *Mathematics* is the *Science* of Quantity, or the *Art* of Reasoning about Things, that are capable of *more* and *less*, and that the most part of the Objects of our Knowledge are such, as *Matter*, *Space*, *Number*, *Time*, *Motion*, *Gravity*, &c. We have but imperfect Ideas of Things without Quantity, and as imperfect a one of Quantity itself without the help of *Mathematics*. All the visible

visible Works of GOD Almighty, are made in *Number*, *Weight*, and *Measure*; and therefore to consider them, we ought to understand *Arithmetic*, *Geometry*, and *Statics*: And the greater Advances we make in those Arts, the more capable we are of considering such things, as are the ordinary Objects of our Conceptions. But this will farther appear from Particulars.

And first, if we consider, to what perfection we now know the Courses, Periods, Orders, Distances, and Proportions of the several great Bodies of the Universe, at least such as fall within our View; we shall have cause to admire the Sagacity and Industry of the *Mathematicians*, and the Power of *Numbers*, and *Geometry*, well apply'd. Let us cast our Eyes backward, and consider *Astronomy* in its Infancy; or rather let us suppose it still to begin; for Instance, a Colony of rude Country People, transplanted into an Island, remote from the Commerce of all Mankind, without so much as the knowledge of the Kalendar, and the Periods of the Seasons, without Instruments to make Observations, or any the least Notion of Observations or Instruments. When is it, we could expect any of their Posterity should arrive at the Art of predicting an Eclipse? Not only so, but the Art of reckoning all Eclipses that are past, or to come, for any number of Years? When is it we could suppose that one of those Islanders, transported to any other Place of the Earth, should be able, by the Inspection of the Heavens, to find how much he were North, or South, East, or West, of his own Island, and to conduct his Ship back thither? For my part, tho' I know this may be, and is daily done by what is known in *Astronomy*; yet when

when I consider the vast Industry, Sagacity, Multitude of Observations, and other extrinsick Things, necessary for such a Sublime Piece of Knowledge, I should be apt to pronounce it impossible, and never to be hoped for. Now we are let so much into the Knowledge of the Machine of the Universe, and Motion of its Parts, by the Rules of this Science, perhaps the Invention may seem easy: But when we reflect, what Penetration and Contrivance were necessary to lay the Foundation of so great and extensive an Art, we cannot but admire its first Inventors; as *Thales Milesius*, who as *Diogenes Laertius*, and *Pliny* say, first predicted Eclipses; and his Scholar *Anaximander Milesius*, who found out the Globous Figure of the Earth, the Equinoctial Points, the Obliquity of the Ecliptic, the Principles of Gnomonics, and made the first Sphere or Image of the Heavens; and *Pythagoras*, to whom we owe the discovery of the true System of the World, and Order of the Planets. Tho' it may be they were assisted by the *Egyptians* and *Chaldeans*. But whoever they were, that first made these bold Steps in this noble Art, they deserve the Praise and Admiration of all future Ages.

Felices Animae, quibus haec cognoscere primis,
Inque Domos superas scandere cura fuit.
Credibile est illos pariter vitiisque jocisque
Altius humanis exseruisse caput.
Non Venus et Vinum sublimia pectora fregit,
Officiumque fori, militiaeque Labor:
Non levis ambitio, perfusaque gloria fuco,
Magnarumque fames sollicitavit opum.
Admovere Oculis distantia Sidera nostris,
Aetheraque ingenio supposuere suo.
Ovid. 1. Fast.

But

But tho' the Induſtry of former Ages had diſcovered the Periods of the great Bodies of the Univerſe, and the true Syſtem, and Order of them, and their Orbits pretty near; yet was there one thing ſtill reſerved for the glory of this Age, and the honour of the *Engliſh* Nation, the grand Secret of the whole Machine; which, now it is diſcovered, proves to be (like other Contrivances of Infinite Wiſdom) ſimple and natural, depending upon the moſt known, and moſt common Property of Matter, *viz. Gravity*. From this the incomparable Sir *Iſaac Newton* has demonſtrated the Theories of all the Bodies of the Solar Syſtem, of all the Primary *Planets*, and their *Secondaries*, and among others, the *Moon* which ſeem'd moſt averſe to Numbers: And not only of the Planets, the ſloweſt of which compleats its Period in leſs than half the Age of a Man, but likewiſe of the Comets, ſome of which its probable, ſpend more than 2000 Years in one Revolution about the Sun; for whoſe Theory he has laid ſuch a Foundation, that After-ages aſſiſted with more Obſervations, may be able to calculate their returns. In a word, the Preceſſion of the Equinoctial Points, the Tides, the unequal Vibration of Pendulous Bodies in different Latitudes, *&c.* are no more a Queſtion to thoſe, that have *Geometry* enough to underſtand, what he has delivered on thoſe Subjects. A Perfection in *Philoſophy*, that the boldeſt Thinker durſt hardly have hoped for; and, unleſs Mankind turn barbarous, will continue the Reputation of this Nation, as long as the Fabric of Nature ſhall endure. After this, what is it we may not expect from *Geometry* join'd to *Obſervations* and Experiments?

The next considerable Object of natural Knowledge, I take to be *Light*. How unsuccessful Enquiries are about this glorious Body without the Help of *Geometry*, may appear from the empty and frivolous Discourses and Disputations of a sort of Men, that call themselves *Philosophers*, whom nothing will serve forsooth, but the Knowledge of the very Nature, and intimate Causes of every Thing; while on the other hand, the *Geometers* not troubling themselves with these fruitless Enquiries about the *Nature* of *Light*, have discovered two remarkable Properties of it, in the Reflection and Refraction of its Beams: and from those, and their Streightness in other Cases, have invented the noble Arts of *Optics*, *Catoptrics*, and *Dioptrics*; teaching us to manage this subtile Body for the improvement of our Knowledge, and useful purposes of Life. They have likewise demonstrated the Causes of several cælestial Appearances, that arise from the Inflexion of its Beams, both in the Heavenly Bodies themselves, and other *Phænomena*, as *Parhelia*, the *Iris*, &c. and by a modern Experiment, they have discovered the Celerity of its Motion. And we have had surprizing Properties of Light, since the great Sir *Isaac Newton* has been pleas'd to gratify the World with his Book of *Light and Colours*.

The *Fluids*, which involve our Earth, *viz.* *Air* and *Water*, are the next great and conspicuous Bodies, that Nature presents to our View, and I think we know little of either, but what is owing to *Mechanics* and *Geometry*. The two chiefest Properties of *Air*, its Gravity and Elastic Force, have been discovered by mechanical Experiments. From thence the decrease of the Air's Density according

according to the increaſe of the Diſtance of the Earth, has been demonſtrated by *Geometers*, and confirmed by Experiments of the Subſidence of the *Mercury* in the *Torricellian Experiment*. From this likewiſe, by Aſſiſtance of *Geometry*, they have determined the Height of the Atmoſphere, as far as it has any ſenſible Denſity; which agrees exactly with another Obſervation of the Duration of the Twilight. *Air* and *Water* make up the Object of the Hydroſtatics, though denominated only from the latter, of which the Principles were long ſince ſettled and demonſtrated by *Archimedes*, in his Book περὶ τῶν Ὀχουμένων, where are demonſtrated the Cauſes of ſeveral ſurprizing *Phænomena* of Nature, depending only on the *Æquilibrium* of *Fluids*, the Relative Gravities of theſe *Fluids*, and of Solids ſwimming or ſinking therein. Here alſo the Mathematicians conſider the different Preſſures, Reſiſtances, and Celerities of Solids moved in *Fluids*; from whence they explain a great many Appearances of Nature, unintelligible to thoſe who are ignorant of *Geometry*.

Next, if we deſcend to the *Animal Kingdom*, there we may ſee the brighteſt Strokes of Divine *Mechanics*. And whether we conſider firſt the *Animal Oeconomy* in general, either in the internal Motion, and Circulation of the Juices forced through the ſeveral Canals by the Motion of the Heart, or their external Motions and the Inſtruments wherewith thoſe are perform'd, we muſt reduce them to Mechanical Rules, and confeſs the neceſſity of the Knowledge of *Mechanics* to underſtand them, or explain them to others. *Borelli*, in his excellent Treatiſe *de motu Animalium*, *Steno* in his admirable *Myologiæ Specimen*, and other Mathematical Men on the one hand, and the nonsensical, unintelligible Stuff that the common Writers on these Subject have fill'd their Books with, on the other, are sufficient Instances to shew how necessary *Geometry* is in such Speculations. The only Organ of an Animal Body, whose Structure and Manner of Operation is fully understood, has been the only one, which the *Geometers* have taken to their Share to consider.

It is incredible, how sillily the greatest and ablest Physicians talked of the Parts of the Eye, and their Use, and of the *Modus Visionis*, before *Kepler* by his *Geometry* found it out, and put it past dispute, tho' they apply'd themselves particularly to this, and valued themselves on it: And *Galen* pretended a particular Divine Commission to treat of it. Nay, notwithstanding the full Discovery of it, some go on in copying their Predecessors, and talk as much *Ungeometrically* as ever. It's true, we cannot reason so clearly of the internal Motions of an Animal Body, as of the external, wanting sufficient *data*, and decisive Experiments: But what relates to the latter (as the Articulation, Structure, Insertion, and Vires of the Muscles) is as Subject to strict Mathematical Disquisition, as any thing whatsoever: And even in the Theory of Diseases, and their Cures, those who talk Mechanically, talk most Intelligibly; which may be the Reason for the Opinion of the ancient Physicians, that *Mathematics* are necessary for the Study of Medicine itself; for which I could bring long Quotations out of their Works. Among the Letters that are ascrib'd to *Hippocrates*, there is one to his Son *Thessalus*, recommending to him the Study of *Arithmetic*, and Geometry, as necessary to Medicine. *Galen*
in

in his Book entituled ὅτι ἄριστος ἰατρὸς καὶ φιλόσοφος, begins thus.

> Οἷόν τι πεπόνθασιν οἱ πολλοὶ τῶν ἀθλητῶν ἐπιθυμοῦντες μὲν Ὀλυμπιονῖκαι γενέσθαι, μηδὲν δὲ πράττειν ὡς τούτου τυχεῖν ἐπιτηδεύοντες, τοιοῦτόν τι καὶ τοῖς πολλοῖς τῶν ἰατρῶν συμβέβηκεν· ἐπαινοῦσι μὲν γὰρ Ἱπποκράτην καὶ πρῶτον ἁπάντων ἡγοῦνται· γενέσθαι δ' αὐτοὺς ἐν ὁμοίοις ἐκείνῳ πάντα μᾶλλον ἢ τοῦτο πράττουσι. οἱ μὲν γὰρ οὐ μικρὰν μοῖραν εἰς ἰατρικήν φησὶ συμβάλλεσθαι τὴν ἀστρονομίαν, καὶ δηλονότι τὴν ταύτης ἡγουμένην ἐξ ἀνάγκης Γεωμετρίαν. οἱ δ' οὐ μόνον αὐτοὶ μετέρχονται τούτων οὐδετέραν, ἀλλὰ καὶ τοῖς μετιοῦσι μέμφονται.

If one of the Reasons of the Ancients for this be now somewhat unfashionable, *viz.* because they thought a Physician should be able to know the Situation and Aspects of the Stars, which they believed had Influence upon Men and their Diseases (and positively to deny it, and say, that they have none at all, is the effect of want of Observation) we have a much better and undoubted one in the Room, *viz.* That *Mathematics* are found to be the best Instrument of promoting natural Knowledge. Secondly, If we consider, not only the animal Œconomy in general, but likewise the wonderful Structure of the different sorts of Animals, according to the different purposes for which they were design'd, the various Elements they inhabit, the several ways of procuring their Nourishment, and propagating their Species, the different Enemies they have, and Accidents they are subject to, here is still a greater need of *Geometry*. It is pity, that the Qualities of an expert *Anatomist* and skilful *Geometer*, have seldom met in the same Person. When such a One shall appear, there is a whole

Terra

Terra incognita of delightful Knowledge to employ his Time, and reward his Industry.

As for the other two Kingdoms; *Borelli* and other *Mathematical Men*, seem to have talked very clearly of *Vegetation*: And *Steno*, another Mathematician, in his excellent Treatise *de Solido intra Solidum naturaliter contento*, has applied this part of Learning very handsomely to *Fossils*, and some other Parts of Natural History. I shall add only one thing more, that if we consider Motion itself, the great Instrument of the Actions of Bodies upon one another; the Theory of it is entirely owing to the *Geometers*, who have demonstrated its Laws both in hard and elastic Bodies; shew'd how to measure its Quantity, how to compound and resolve the several Forces, by which Bodies are agitated, and to determine the *Lines* which those compound Forces make them describe; of such Forces, Gravity, being the most constant and uniform, affords a great variety of useful Knowledge, in considering several Motions that happen upon the Earth, *viz.* As to the free Descent of heavy Bodies, the Curve of Projectiles; the Descent and Weight of the heavy Bodies when they lye on inclined Planes; the Theory of the Motion of Pendulous Bodies, *&c.* all which are very ingeniously and methodically treated of by the imcomparable Mathematician Mr. *Thomas Simpson*, who has exceeded all Men (in Mathematical Sciences) since Sir *Isaac Newton*.

From what I have said, I shall draw but one *Corollary*, that a natural Philosopher without Mathematics is a very odd sort of a Person, that reasons about things that have *Bulk*, *Figure*, *Motion*, *Number*, *Weight*, &c. without *Arithmetic*,

tic, *Geometry*, *Mechanics*, *Statics*, &c. I must needs say I have the last Contempt for those Gentlemen, that pretend to explain how the Earth was framed, and yet can hardly Measure an Acre of Ground upon the Surface of it. And as the Philosopher speaks, *Qui repente pedibus illotis ad Philosophos divertunt, non hoc est satis, quod sint omnino* ἀθεώρητοι, ἄμουσοι, γεωμέτρητοι, sed legem *etiam dant, quâ philosophari discant.*

The Usefulness of *Mathematics* in several other Arts and Sciences is fully as plain. They were look'd upon by the antient Philosophers, as the Key to all Knowledge. Therefore *Plato* wrote upon his School, *viz.*

Οὐδεὶς ἀγεωμέτρητος εἰσίτω. *i. e.*

Let none unskilled in Geometry enter.

And *Xenocrates* told one ignorant in *Mathematics*, who desired to be his Scholar, that he was fitter to card Wool, λαβὰς γὰρ οὐκ ἔχεις φιλοσοφίας, *you want the handle of Philosophy*, viz. *Geometry*. There is no understanding the Works without it. *Theo Smyrnus* has wrote a Book entituled, an Explanation of those things in *Mathematics*, that are necessary for the reading of *Plato*. *Aristotle* illustrates his Precepts and other Thoughts by Mathematical Examples, and that not only in *Logic*, &c. but even in *Ethics*; where he makes use of Geometrical, and Arithmetical Proportion, to explain Commutative and Distributive Justice.

Every Body knows, that *Chronology* and *Geography* are indispensable Preparations for History; A relation of Matter of Fact, being a very lifeless insipid thing without the Circumstances of Time and Place. " Nor is it sufficient for one,

that

that would understand things thoroughly, that he knows the Topography, where such a Place lies, with those of the near adjacent Places, and how these lie in respect of one another, but it will become him likewise to understand the Scientifical Principles of the Art; that is, to have a true Idea of a Place, we ought to know the relation it has to any other Place, as to the Distance and Bearing, its Climate, Heat, Cold, Length of Days, *&c.* which things do much enliven the Reader's Notion of the very Art itself. Just so it is necessary to know the Technical or Doctrinal Part of Chronology, if a Man would be throughly skill'd in History, it being impossible, without it, to unravel the Confusion of Historians. I remember the late Dr. *Halley* has determined the Day and Hour of *Julius Cæsar*'s landing in *Britain*, from the Circumstances of his Relation. And every Body knows how great use Mr. *Dodwell* has made of the calculated Times of Eclipses, for settling the Times of great Events, which before were, as to this *essential* Circumstance, almost fabulous.

Both *Chronology* and *Geography*, and also the Knowledge of the *Sun* and *Moon*'s Motions, so far as they relate to the Constitution of the Kalendar and Year, are necessary to a Divine, and how sadly some otherwise Eminent have blundered, when they meddled with things that relate to *these*, and border on *them*, is too apparent.

Nobody, I think, will question the Interest, that *Mathematics* have in *Painting*, *Music* and *Architecture*, which are all founded on Numbers. Perspective, and the Rules of Light and Shadows are owing to *Geometry* and *Optics:* And I think those two comprehend pretty near the whole Art of *Painting*, except *Decorum* and *Ordinance*, which are only a due Obſervance of the Hiſtory and Circumſtances of the Subject you repreſent. For by Perſpective may be underſtood the Art of deſigning the Out-lines of your Solid, whether that be a *Building*, *Landskip*, or *Animal*; and the Draught of a Man is really as much the Perſpective of a Man, as the Draught of a Building is of a Building; tho' for particular Reaſons, as becauſe it conſiſts of more crooked Lines, *&c.* it is hard to reduce the Perſpective of the former to the ordinary, eſtabliſhed Rules.

If *Mathematics* had not reduced *Muſic* to a regular Syſtem, by contriving its *Scales*, it had been no Art, but Enthuſiaſtic Rapture, left to the roving Fancy of every Practitioner. This appears by the extraordinary Pains which the Ancients have taken to fit Numbers to three Sorts of *Muſic*, the *Diatonic*, *Chromatic*, and *Enharmonic*: Which, if we conſider with their Nicety in diſtinguiſhing their ſeveral *Modes*, we ſhall be apt to judge they had ſomething very fine in their *Muſic*, at leaſt for moving the Paſſions with ſingle Inſtruments, and Voices. But Muſic had been imperfect ſtill, had not Arithmetic ſtepp'd in once more, and *Guido Aretinus*, by inventing the Temperament, making the Fifth falſe by a certain determined Quantity, taught us to tune our *Organs*, and intermix all the three kinds of the Ancients; to which we owe all the regular and noble Harmony of our modern Muſic.

As for *Civil Architecture* (of Military I ſhall ſpeak afterward) there is hardly any Part of Mathematics but is ſome Way ſubſervient to it. *Geometry* and *Arithmetic*, for the due Meaſure of the ſeveral Parts of a *Building*, the *Plans*, *Models*, Computation of Materials, Time and Charges; for ordering right its Arches and Vaults, that they may be both firm and beautiful: *Mechanics*, for its Strength and Firmneſs, tranſporting and raiſing Materials; and *Optics* for the Symmetry and Beauty. And I would not have any aſſume the Character of an *Architect*, without a competent Skill in all theſe. You ſee that *Vitruvius* requires theſe and many more for making a compleat *Architect*. I muſt own, that ſhould any one ſet up to practice in any of theſe fore-mentioned Arts, furniſhed only with his *Mathematical* Rules, he would produce but very clumſy Pieces. He that ſhould pretend to draw by the Geometrical Rules of Perſpective, or compoſe *Muſic* meerly by his Skill in Harmonical Numbers, would ſhew but aukward Performances. In thoſe compoſed Subjects, beſides the ſtiff Rules, there muſt be Fancy, Genius, and Habit. Yet nevertheleſs theſe Arts owe their Being to *Mathematics*, as laying the Foundation of their Theory, and affording them Precepts, which being once invented, are ſecurely rely'd upon by Practitioners. Thus many deſign, that know not a Tittle of the Reaſon of the Rules they practice by; and many, no better qualified in their Way, compoſe *Muſic* better perhaps than he could have done that invented the *Scale* and the *Numbers* upon which their Harmony is founded. As *Mathematics* laid the Foundation of theſe Arts, ſo they muſt improve them; and thoſe that would invent muſt be ſkilled in Numbers. Beſides, it is fit a Man ſhould know the true Grounds and Reaſons of what he ſtudies; aud he that does ſo, will certainly practice in his

Art

Art with greater Judgment and Variety, where the ordinary Rules fail him.

I proceed now to ſhew the more immediate Uſefulneſs of *Mathematics* in *Civil Affairs*. To begin with *Arithmetic*, it were an endleſs Taſk to relate its ſeveral Uſes in public and private Buſineſs. The Regulation and quick Diſpatch of both, ſeem intirely owing to it. The Nations that want it are altogether barbarous, as ſome *Americans*, who can hardly reckon above twenty. And I believe it would go near to ruin the Trade of the Nation, were the eaſy Practice of *Arithmetic* aboliſhed: For Example, were the Merchants and Tradeſmen obliged to make uſe of no other than the *Roman* Way of Notation by Letters, inſtead of our preſent. And if we ſhould feel the Want of our *Arithmetic* in the eaſieſt Calculations, how much more in thoſe that are ſomething harder, as Intereſt, Simple and Compound, Annuities, *&c.* in which it is incredible how much the ordinary Rules and Tables influence the Diſpatch of Buſineſs. *Arithmetic* is not only the great Inſtrument of private Commerce, but by it are (or ought to be) kept the public Accounts of a Nation: I mean thoſe that regard the whole State of a Commonwealth, as to the Number, Fructification of its People, Increaſe of Stock, Improvement of Lands and Manufactures, Balance of Trade, Public Revenues, Coinage, Military Power by Sea and Land, *&c.* Thoſe that would judge or reaſon truly, about the State of any Nation, muſt go that Way to work, ſubjecting all the fore-mentioned Particulars to Calculation. This is the true, *political* Knowledge. In this Reſpect, the Affairs of a Commonwealth differ from thoſe of a private

Family only in the Greatness and Multitude of Particulars that make up the Accounts. *Machiavel* goes this Way to work in his Account of different Estates. What Sir *William Petty*, Dr. *Halley*, and several others of our Countrymen, have wrote in *Political Arithmetic*, does abundantly shew the Pleasure and Usefulness of such Speculations.

Lastly, Numbers are applicable even to such Things as seem to be governed by no Rule; I mean such as depend on Chance. The Quantity of Probability, and Proportion of it, in any two proposed Cases, being subject to Calculation as much as any Thing else. Upon this depend the Principles of Game, which has been thoroughly handled by the sagacious Mathematician, Mr. *Thomas Simpson*, in his Doctrine of Chances. We find Sharpers know enough of this to cheat some Men that would take it very ill to be thought Bubbles: And one Gamester exceeds another, as he has a greater Sagacity and Readiness in calculating his Probability to win or lose in any proposed Case. To understand the *Theory* of *Chance* thoroughly, requires a great Knowledge of Numbers, and a pretty competent one in *Algebra*.

The several Uses of *Geometry* are not much fewer than those of Arithmetic. It is necessary for ascertaining of Property both in *Planes* and *Solids*, or in *Surveying* and *Gauging*. By it Land is sold by Measure as well as Cloth; Workmen are paid the due Price of their Labour, according to the Superficial or Solid Measure of their Work: And the Quantity of Liquors determined for a due Regulation of their Price and Duty. All which do wonderfully conduce to the easy

Dispatch

Diſpatch of Buſineſs, and the preventing of Frauds and Controverſies. I need not mention the meaſuring of Diſtances, laying down of Plans and Maps of Countries, in which we have daily Experience of its Uſefulneſs. Theſe are ſome Familiar Inſtances of Things to which *Geometry* is ordinarily apply'd.

From *Aſtronomy*, we have the regular Diſpoſition of our Time, in a due Succeſſion of Years, which are kept within their Limits, as to the Return of the Seaſons, and the Motion of the Sun. This is no ſmall Advantage for the due Repetition of the ſame Work, Labour, and Actions. For many of our public, private, military and Country Affairs, Appointments, *&c.* depending on the Products of the Ground, and they on the Seaſons, it is neceſſary that the Returns of them be adjuſted pretty near to the Motion of the Sun; and we ſhould quickly find the Inconveniency of a *Vague*, undetermined Year, if we uſed that of the *Mahometans*, whoſe Beginning, and every Month, wanders through all the Days of ours or the Solar Year, which ſhews the Seaſons. Beſide, the adjuſting of the Moon's Motion to the Sun's is required, for the decent Obſervation and Celebration of our *Church Feaſts* and *Faſts* according to the ancient Cuſtom and Primitive Inſtitution; and likewiſe for the Knowledge of the Ebbing and Flowing of the Tides, the Spring and Neap Tides, Currents, *&c.* So that whatever ſome People may think of an *Almanack*, where all theſe are ſet down, it is oftentimes the moſt uſeful Paper that is publiſhed the ſame Year with it; Nay, the Nation could better ſpare all the Voluminous Authors in the *Term Catalogue* than that ſingle Sheet.

Beſides,

Besides, without a regular *Chronology*, there can be no certain History; which appears by the Confusion amongst Historians before the right Disposition of the Year, and at present among the *Turks*, who have the same Confusion in their History as in their Kalendar. Therefore a Matter of such Importance might well deserve the Care of the *Great Emperor*, to whom we owe our present Kalendar; who was himself a Proficient in *Astronomy*. *Pliny* has quoted several Things from his Books of the *Rising* and *Setting* of the Stars. Lib. 18. c. 25, 26. and *Lucan* makes him say,

——*Media inter Prælia semper*
Stellarum, Cœlique plagis, superisque vacavi.

The Mechanics have produced so many useful Engines subservient to Conveniency, that it would be a Task too great to relate the several Sorts of them; some of them keep Life itself from being a Burden. If we consider such as are invented for raising Weights, and are employ'd in Building, and other great Works, in which no Impediment is too great for them; or *Hydraulic* Engines for raising of Water, serving for great Use and Comfort to Mankind, where they have no other Way to be supply'd readily with that necessary Element; for such as by making Wind and Water work for us, save Animal Eorce, and great Charges, and perform those Actions which require a vast Multitude of Hands, and without which every Man's Time would be too little to prepare his own Aliment and other Necessaries; or those *Machines* that have been invented by Mankind for Delight and Curiosity,

Curiosity, imitating the Motions of Animals, or other Works of Nature, we shall have Reason to admire and extoll so excellent an Art. What shall we say of the several Instruments which are contriv'd to measure Time? We should quickly find the Value of them, if we were reduced to the Condition of those barbarous Nations that want them. The *Pendulum Clock*, invented and compleated by that famous *Mathematician* Monsieur *Hugens*, is an useful Invention. Is there any thing more wonderful than several *Planetary Machines*, which have been invented to shew the Motions of the Heavenly Bodies, and their Places at any Time? Of which the most ingenious, according to the exactest Numbers, and true System, was made by the same M. *Hugens*, to which we may very justly apply *Claudian's* noble Verses upon that of *Archimedes*.

Jupiter *in parvo cum cerneret Aethera vitro,*
Risit, et ad superos talia dicta dedit.
Huccine mortalis progressa potentia curæ?
Jam meus in fragili luditur orbe labor.
Jura Poli, rerumque fidem, legesque Deorum
Ecce Syracusius *transtulit arte senex.*
Inclusus variis famulatur Spiritus astris,
Et vivum certis motibus urget Opus.
Percurrit proprium mentitus Signifer Annum,
Et simulata nova Cynthia *mense redit.*
Jamque suum volvens audax industria mundum
Gaudet, et humanâ sidera Mente regit.
Quid falso insontem tonitru Salmonea miror?
Aemula naturæ parva reperta manus.

Here I ought to mention the *Sciatherical* Instruments, for Want of which there was a Time when

when the *Grecians* themselves were forced to measure the Shadow, in order to know the Hour of the Day. To this ought to be referr'd *Spheres*, *Globes*, *Astrolabes*, Projections of the Sphere, *&c.* These are such useful and necessary Things, that alone may recommend the Art by which they are made. For by these we are able in our Closet to judge of the Celestial Motions, and to visit the utmost distant Places of the Earth, without the Fatigue and Danger of Voyages; to determine concerning their Distance, Situation, Climate, Nature of the Seasons, Length of their Days, and their Relation to the Celestial Bodies, as much as if we were Inhabitants. To these I might add those Instruments which the *Mathematicians* have invented to execute their own Precepts, for making *Observations* either by Sea or Land, *Surveying*, *Gauging*, &c.

The *Catoptrics* and *Dioptrics* furnish us with Variety of useful Inventions, both for the promoting of Knowledge and the Conveniences of Life; whereby Sight, the great Instrument of our Perception, is so much improved, that neither the Distance, nor the Minuteness of the Object, are any more Impediments to it. The *Telescope* is of so vast Use, that besides the delightful and useful Purposes it is apply'd to here below, as the descrying Ships, and Men, and Armies at a Distance, we have by its Means discovered new Parts of the Creation, fresh Instances of the surprizing Wisdom of the adorable *Creator*; we have by it discovered the *Satellites* of *Jupiter*, the *Satellites* and *Ring* of *Saturn*, the Rotation of the Planets about their own Axes, besides other Appearances, whereby the System of the World is made plain to *Sense*, as

it

it was before to *Reason*. The *Telescope* has also improved the Manner of *Astronomical Observations*, and made them much more accurate, than it was possible for them to be before. And these Improvements in *Astronomy* have brought along with them (as ever) correspondent Improvements in *Geography*. From the Observation of *Jupiter's Satellites*, we have a ready Way to determine the Longitude of Places on the Earth. On the other hand, the *Microscope* has not been less useful in helping us to the Sight of such Objects, as by their Minuteness escape our naked Eye. By it Men have pursued Nature into its most retired Recesses; so that now it can hardly any more hide its great Mysteries from us. How much have we learned by the Help of the *Microscope*, of the Contrivance and Structure of Animal and Vegetable Bodies, and the Composition of Fluids and Solids? But if these *Sciences* had never gone further than by their single *Specula* and *Lentes* to give those surprizing Appearances of Objects and their Images, and to produce Heat inimitable by our hottest Furnaces, and to furnish infallible, easy, cheap, and safe Remedies for the Decay of our Sight, arising commonly from old Age, and for Purblindness; they had merited the greatest Esteem, and invited to the closest Study: Especially, if we consider, that those who naturally are almost blind, and either know not their nearest Acquaintance at the Distance of a Room's Breadth, or cannot read in order to pass their Time pleasantly, are, by Glasses adapted to the Defect of their Eyes, set on a Level again with those that enjoy their Eye-sight best, and that without Danger, Pain or Charge.

Again, *Mathematics* are highly ſerviceable to a Nation in *Military Affairs*. I believe this will be readily acknowledged by every Body. The Affairs of War take in Number, Space, Force, Diſtance, Time, *&c.* (Things of *Mathematical* Conſideration) in all its Parts, in *Tactics*, *Castrametation*, *Fortifying*, *Attacking* and *Defending*. The Ancients had more Occaſion for *Mechanics* in the Art of War than we have: Gun-powder readily producing a Force far exceeding all the Engines, they had contrived for Battery. And this I reckon has loſt us a good Occaſion of improving our *Mechanics*: The Cunning of Mankind never exerting itſelf ſo much, as in their Arts of deſtroying one another. But, as Gunpowder has made *Mechanics* leſs ſerviceable to War; it has made *Geometry* more neceſſary; there being a Force or Reſiſtance in the due Meaſures and Proportions of the Lines and Angles of a Fortification, which contribute much towards its Strength. This Art of *Fortification* has been much ſtudied of late, but I dare not affirm that it has attained its utmoſt Perfection. And tho' where the Ground is regular, it admits but of ſmall Variety, the Meaſures being pretty well determined by *Geometry* and Experience, yet where the Ground is made up of natural *Strengths* and *Weakneſſes*, it affords ſome Scope for thinking and Contrivance. But there is another much harder Piece of *Geometry*, which Gunpowder has given us Occaſion to improve, and that is the Doctrine of Projectiles, whereupon the Art of Gunnery is founded; and I think no One has given neater Inveſtigations of the Fundamental Principles of Projectiles, than the acute and ingenious Mathematician, Mr.

Thomas

Thomas Simpson, from Page 143 to 163, in his excellent Treatise of Fluxions.

As for *Tactics* and *Castrametation*, *Mathematics* retain the same Place in them as ever. And some tolerable Skill in these is necessary for Officers as well as for Engineers. An Officer that understands Fortification, will *cæteris paribus* much better defend his Post, as knowing wherein its Strength consists, or make use of his Advantages and Disadvantages in defending and attacking, how to make the best of his Ground, *&c.* And hereby can do truly more Service than another of as much Courage, who, for want of such Knowledge, it may be, throws away himself, and a Number of brave Fellows under his Command; and it is well, if the Mischief reaches no further. As for a competent Skill in *Numbers*, it is so necessary for *Officers*, that no Man can be safely trusted with a Company that has it not. And I dare appeal to all the Nations in *Europe*, whether, *cæteris paribus*, Officers are not advanced in Proportion to their Skill in *Mathematical Learning*, except that sometimes great Names and Quality carry it; but still so as that the Prince depends upon a Man of *Mathematical Learning*, that is put as Director to the *Quality*, when that Learning is wanting in it.

Lastly, NAVIGATION, which is made up of *Astronomy* and *Geometry*, is so noble an Art, and to which Mankind owes so many Advantages, that upon this single Account those excellent Sciences deserve most of all to be study'd, and merit the greatest Encouragement from a Nation that owes to it both its Riches and Security. And not only the common Art of *Navi-*

gation depends on *Mathematics*, but whatever Improvements ſhall be made in the *Architectura Navalis*, or building of Ships, whether they are deſigned for Merchant Ships, or Ships of War, whether ſwift running, or bearing a great Sail, or lying near the Wind, be deſired, theſe muſt all be the Improvements of *Geometry*. Ship Carpenters indeed are very induſtrious; but in theſe Things they acknowledge their Inability, confeſs that their beſt Productions are the Effects of Chance, and implore the *Geometer*'s Help. Nor will common *Geometry* do the Buſineſs, it requires the moſt abſtruſe to determine the different Sections of a Ship, according as it is deſign'd for any of the aforeſaid Ends.

The great Objection that is made againſt the Neceſſity of *Mathematics* in the forementioned great Affairs of *Navigation*, the Art *Military*, &c. is, that we ſee thoſe Affairs are carried on and managed by ſuch as are not great Mathematicians, as Seamen, Engineers, Surveyors, Gaugers, Clockmakers, Glaſs-Grinders, &c. and that the *Mathematicians* are commonly ſpeculative, retir'd, ſtudious Men; that are not for an active Life and Buſineſs, but content themſelves to ſit in their Studies, and pore over a Scheme or Calculation. To which there is this plain and eaſy Anſwer. The Mathematicians have not only invented and order'd all the Arts above mentioned, by which thoſe grand Affairs are managed, but have laid down Precepts, contrived Inſtruments and Abridgments ſo plainly, that common Artificers are capable of practiſing by them, tho' they underſtand not a Tittle of the Grounds on which the Precepts are built. And in this they have conſulted the Neceſſities of

Mankind.

Mankind. Thoſe Affairs demand ſo great a Number of People to manage them, that it is impoſſible to breed ſo many good, or even tolerable *Mathematicians*. The only Thing then to be done was, to make their Precepts ſo plain, that they might be underſtood and practiſed by a Multitude of Men. This will beſt appear by Examples. Nothing is more ordinary than Diſpatch of Buſineſs by common *Arithmetic*, by the Tables of *Simple* and *Compound Intereſt*, *Annuities*, &c. Yet how few Men of Buſineſs, nay pretended Teachers themſelves, underſtand the Reaſons of common *Arithmetic*, or the Contrivance of thoſe Tables, now they are made; but ſecurely rely on them as true. They were the good and thorough Mathematicians that made thoſe Precepts ſo plain, and calculated thoſe Tables, that facilitate the Practice ſo much. Nothing is more univerſally neceſſary than the meaſuring of Plains and Solids: And it is impoſſible to breed ſo many good *Mathematicians*, as that there may be one that underſtands all the *Geometry* requiſite for Surveying and Meaſuring of *Priſms* and *Pyramids*, and their Parts, and Meaſuring *Fruſtrums* of *Conoids* and *Spheroids* in every Market Town, where ſuch Work is neceſſary. The Mathematicians therefore have inſcrib'd ſuch Lines on their common Rulers, and ſlipping Rulers, and adapted ſo plain Precepts to them, that every Country Carpenter and Gauger can do the Buſineſs accurately enough; tho' he knows no more of thoſe Inſtruments, Tables and Precepts he makes uſe of, than a Hobby-Horſe. So in *Navigation*, it is impoſſible to breed ſo many good Mathematicians, as would be neceſſary to ſail the hundredth Part of the Ships of the Nation. But the

the *Mathematicians* have laid down so plain and distinct Precepts, calculated necessary Tables, and contrived convenient Instruments, so that a Sea-Man, that knows not the Truths on which his Precepts and Tables depend, may practice safely by them. They resolve Triangles every Day, that know not the Reason of any one of their Operations. Seamen, in their Calculations, make use of Artificial Numbers or *Logarithms*, that know nothing of their Contrivance. And indeed all those great Inventions of the most famous *Mathematicians*, had been almost useless for those common and great Affairs, had not the Practice of them been made easy to those who cannot understand them. From hence it is plain, that it is to those *speculative*, *retir'd Men* we owe the Rules, the Instruments, the Precepts for using them, and the Tables which facilitate the Dispatch of so many great Affairs, and supply Mankind with so many Conveniencies of Life. They were the Men that taught the World to apply *Arithmetic*, *Astronomy* and *Geometry* to *Sailing*, without which the Needle would be still useless. Just the same Way in the other Parts of *Mathematics*, the Precepts that are practised by Multitudes without being understood, were contrived by some few great *Mathematicians*; as for Instance,

"How few understand this *allow'd for Truth* "*in the known* common Way of proving Multi-"plication and Division by casting away the "Nines, that

PROPO-

PROPOSITION.

" The Number 9 has that Property, that any " Number whatſoever divided by it, ſhall leave " the ſame Remainder, as if the Sum of the " Figures compoſing the ſaid Number, were di- " vided by 9.

" For the clear Demonſtration of this Pro- " poſition, I ſhall premiſe two ſelf-evident

LEMMA's.

1: " The Local Value of any Figure is equal " to the Rectangle of its ſimple Value, and " the Denomination of its Place.

LEMMA.

2. " To multiply or divide one Number by " another, is in Concluſion the ſame Thing with " multiplying or dividing reſpectively the Sum " of the Parts of the former by the latter."

Since then it has been ſhewn how much Mathematics improve the Mind, how ſubſervient they are to other Arts, and how immediately uſeful to the Commonwealth; there needs no other Arguments or Motives to a Government to encourage them. This is the natural Concluſion from theſe Premiſes. *Plato*, in his *Republic*, takes care, *That whoever is to be educated for Magiſtracy, or any conſiderable Poſt in the Commonwealth, may be inſtructed firſt in* Arithmetic, *then in* Geometry, *and thirdly in* Aſtronomy.

The

The present Lords Commissioners of the Admiralty know that there are still two great *Desiderata* in Navigation, to wit, *The Theory of the Variation of the Magnetical Needle*, and *a Method of finding out the Longitude of any Place*, that may be practicable at Sea by Seamen.

And now what can I say more to recommend the Study of the *Mathematicks?* It not only makes a Man of Quality and Estate his whole Life more illustrious and more useful for all Affairs, Ισοκρ. δ' μελέτω σοὶ ὦ παῖ Γεωμετρικῆς κ̀ Αειθμήσιος· ἐ γδ μόνον, σέο καὶ τὸν βίον εὐκλέα καὶ ἐπὶ πολλὰ χρήσιμον ἐς ἀνθρωπίνην μοῖραν ἐπιτελέσει, ἀλλὰ καὶ τὴν ψυχὴν ὀξυτέραν τε κ̀ τηλαυγεστέραν· But in Particular, it is the best Companion for a Country Life. Were this once become a fashionable Study (and the *Mode* exercises its Empire over Learning as well as other Things) it is hard to tell how far it might influence the Morals of our Nobility and Gentry, in rendering them serious, diligent, curious, taking them off from the more fruitless and airy Exercises of the Fancy which they are apt to run into.

The only Objection I can think of, that is brought against these Studies, is, that Mathematics require a particular Turn of Head, and a happy Genius that few People are Masters of, without which all the Pains bestowed upon the Study of them are vain. They imagine, that a *Man must be born a Mathematician.* I answer, that this *Exception* is common to *Mathematics*, and other Arts. That there are Persons that have a particular Capacity and Fitness to one more than another, every Body owns: And from Experience I dare say, it is not in any higher Degree true concerning Mathematics than the others.

A Man

A Man of good Senſe and Application is the Perſon that is by Nature fitted for them, eſpecially if he begins betimes; and if his Circumſtances have been ſuch, that this did not happen by prudent Direction, the Defect may be ſupply'd as much as in any Art whatſoever. The only Advantage this Objection has, is, that it is on the Side of Softneſs and Idleneſs, thoſe powerful Allies.

There is nothing further remains, but that I could wiſh they were practiſed more in our great Schools, to inſtruct the brave, heroic Minds of our *Engliſh* Youth in them, rather than ſtupify and debauch their Morals and Underſtanding by that nonſenſical Ribaldry, the effeminate, foppiſh, *French* Language, which is got to ſuch a Cuſtom amongſt us now a-days, that MASTER muſt have his *Monſieur* forſooth, and ſo the poor School-maſter is *forced* to teach that which he is himſelf a Foreigner to. Let me therefore recommend the Study of the *Greek* and the old brave *Roman*'s Diſcipline, with Philoſophy, *i. e.* Mathematical Studies: But perhaps,

Græcum eſt, non eſt legi,

A Saying in all Reſpects fit enough for the Times of Darkneſs, and Monkiſh Ignorance, may, I am afraid, without much Impropriety, be applied to the preſent Age: The great Indifference, or rather the general Averſion, I have obſerved in theſe latter Years to the *Greek* Tongue, has ſuggeſted to me the following Thoughts. If we look back into Antiquity, and trace the liberal Sciences up to their Source, we ſhall find perhaps the firſt Dawnings of Learning amongst the *Egyptians* and *Chaldeans*. Its next Step was into *Syria* and *Phœnicia*, but here its Advancement was but small, its Progress slow, and its Improvements not very considerable. But when it had extended itself as far as *Greece*, then it began to increase and flourish. Here it met with universal Encouragement, was cultivated with wonderful Success, and grew up to Maturity. Then arose with unusual Splendor the City *Athens*, that illustrious Patroness of Letters, and Metropolis of the learned World: Then were those celebrated Academies established, and those famous Schools of *Pythagoras*, *Socrates*, *Zeno*, *Plato*, *Aristotle*, instituted; which have been the inexhausted Treasures of Philosophy to all Posterity. From hence proceeded the learned and eminent Heroes of Antiquity, that have done Honour to human Nature, and left such Tracts of Glory behind them, as distinguish the Years in which they acted their Parts from the ordinary Course of Time. From this Part of the World, notwithstanding all the Fury and Opposition of Ignorance and Barbarity, have descended down to us those elaborate and excellent Writings, which have been the perfect Copies of whatever is great or noble amongst us. *How can we then despise those glorious Models?*

That no Exceptions can justly be alledged against the Matter of the *Greek* Tongue, is very evident, first, from the general Encouragement that was given to the *Grecian* Philosophers, who being invited over to *Rome*, and resorting thither, brought away with them a great Share of the Politeness and refined Arts of their Country: Witness the famous *Polybius*, *Carneades*, *Diogenes*, *Critolaus*, and others, whose eloquent Discourses

had

had even altered the very Nature and Genius of the *Roman* Youth. For they were ſo enamour'd with Philoſophy, and heard its Lectures with ſuch Pleaſure, that at length they ſhew'd as much Application in their Purſuits after Knowledge, as ever they had done before in the Exerciſe of Pomp and War. Secondly, from the vaſt Improvements that have been made on thoſe ancient Foundations, not only in the *Roman* Empire, but alſo in this and moſt other Nations of *Europe*.

By imitating thoſe bright Examples, by tranſcribing thoſe Originals, and as it were tranſlating *Athens* into herſelf, *Rome* at length became the Seat of Learning, as well as of Empire. Thus, by Reflection, ſhone forth with unparallel'd Luſtre that Meridian of the elegant Sciences the great *Auguſtan* Age. From whence did we derive thoſe Streams of Oratory, Poetry, Hiſtory, Philoſophy, and, in a Word, the whole Circle of Arts and Sciences, but from the firſt Authors and Improvers of them, the Schools of *Greece?* Can we then look upon the Cauſes of ſuch valuable Bleſſings to our Nation with an Eye of Indifference? Shall the lofty *Homer* be buried in Oblivion, and the great *Iliad* be no more? Shall the eloquent *Demoſthenes*, the inimitable *Pindar*, the great *Sophocles*, *Euripides*, *Thucydides*, *Ariſtotle*, lie neglected and forgotten? Shall the Divine *Plato* be loſt for ever? Shall one of the nobleſt of the learned Languages be defaced and obliterated for a little mercenary *French?* No; for if ſuch an unhappy Change as this ſhould ever ſucceed, what muſt be the fatal Conſequence, but a Return of that Night of Ignorance and Error we formerly labour'd under; and that we must of Necessity degenerate again into horrid Barbarity?

Another forcible Argument for maintaining and supporting the *Greek* Tongue, is the Study of the Scriptures and primitive Fathers: Whenever this Language is out of Date, our Religion must of Consequence expire along with it, and be extinguished.

Posterity wisely regulates the Recompences due to learned Men, which puts them on Foot with the greatest Princes; their Glory shines out three thousand Years after they have render'd up their Spirit, and cannot be eclipsed by the Fame of the greatest Heroes. *Homer* is as well known to the Learned as *Achilles*, and the Name of *Virgil* as renowned as that of *Augustus*; and therefore let us prize *Horace* and *Virgil* in our Days, as they were in the Courts of *Augustus*.

Si quia Graecorum sunt antiquissima quaeque
Scripta vel optima; Romani *pensantur eadem*
Scriptores Trutina.———

I shall only add one general Head more upon this Subject, and thence make a Transition to my intended Work.

This Life is a public Theatre, on which Men are to act their Parts. A Thirst after Glory, and an Emulation of whatever is great and excellent, is implanted in our Minds to quicken our Pursuits after real Grandeur, and to enable them to approach, as near as our finite Abilities will admit, to Divinity itself. Upon these Principles we account for the vast Stretch and Penetration of the human Understanding; to these we ascribe the Labours of Men of

Genius;

Genius; and by the Predominancy of them in our Minds, aſcertain the Succeſs of our Attempts. In the ſame Manner we account for that Turn in the Mind, which biaſſes us to admire more what is great and uncommon, than what is ordinary and familiar, however uſeful.

Yet the telling us we were born to purſue what is great, without informing us what is ſo, would avail but little. *Ars Mathematica* declares for a cloſe and attentive Examination of all Things. Outſides and Surfaces may be ſplendid and alluring, yet nothing be within deſerving our Applauſe. He that ſuffers himſelf to be dazzled with a gay and gaudy Appearance, will be betray'd into Admiration of what the Wiſe contemn; his Purſuits will be levelled at Wealth and Power, and high Rank in Life, to the Prejudice of his inward Tranquillity, and perhaps the Wreck of his Virtue. The Pageantry and Pomp of Life will be regarded by ſuch a Perſon as true Honour and Glory, and he will neglect the nobler Acquiſitions, which are more ſuited to the Dignity of his Nature, which alone can give Merit to Ambition, and centre in ſolid and ſubſtantial Grandeur.

The Mind is the Source and Standard of whatever can be conſider'd as great and illuſtrious in any Light: From this our Actions and our Words muſt flow, and by this muſt they be weighed; we muſt think well before we can act or ſpeak as we ought; and it is the inward Vigor of the Soul, tho' variouſly exerted, which forms the Philoſopher, Poet, or Orator. Yet this inward Vigor is chiefly owing to the Bounty of Nature, is cheriſhed and improved by Education, but cannot reach Maturity without other concurrent Causes, such as *public* Liberty, and the strictest Practice of Virtue.

That the Seeds of a great Genius in any kind must be implanted within, and cherished and improved by Education, are Points in which the whole World agrees: But the Importance to Liberty in bringing it to Perfection, may perhaps be more liable to debate. *Ars Mathematica* is clear on the Affirmative Side. The Scope of the *Ars Mathematica* is this; that Genius can never exert itself, or rise to Sublimity, where Virtue is neglected, and the Morals are depraved. Men of the finest Genius which have hitherto appeared in the World, have been for the most part not very defective in their Morals, and less in their Principles. I am sensible there are Exceptions to this Observation, but little to the Credit of the Persons, since their Works become the severest Satyrs on themselves, and the manifest Opposition between their Thought and Practice detracts its Weight from the one, and marks out the other for public Abhorrence.

An inward Grandeur of Soul is the common Centre, from whence every Ray of Sublimity, either in Thought, or Action, or Discourse, is darted out: For all Minds are no more of the same Complexion, than all Bodies of the same Texture. In the latter Case, our Eyes would meet only with the same Uniformity of Colour in every Object. In the former, we should be all Orators or Poets, all Philosophers, or all Blockheads. This would break in upon that useful and beautiful Variety, with which the Author of Nature has adorned the rational, as well as the material Creation. There is in every Mind a Tendency, tho' perhaps differently

inclin'd

inclin'd, to what is great and excellent. Happy they, who know their own peculiar Bent, who have been bleſſed with Opportunities of giving it the proper Culture and Poliſh, and are not cramped or reſtrained in the Liberty of ſhewing and declaring it to others. There are many fortunate Concurrences, without which we cannot attain to any Quickneſs of Taſte or Reliſh for the ſublime Branches of Learning.

ALGEBRA.

ALGEBRA

IS the Art of abstract Reasoning upon Quantity by indefinite and general Representations, in order to solve Problems, invent Theorems, and demonstrate both: It is a Science of Universal Quantity, which properly implies these three Things.

First, *Analysis*, which teaches us to solve Questions, and to demonstrate Theorems, by enquiring into the Bottom, into the Fundamental Constitution and Nature of the Thing, and in this Sense Analytical Demonstrations are opposed to Synthetical ones: The Ancients had some Knowledge of this Art, but kept it concealed, whose Invention *Theo* ascribes to *Plato*, and he defined it (according as *Vieta* renders it) *assumpti quæsiti tanquam concessi per Consequentia ad verum concessum*, *i. e.* a taking of that as granted or confessed, which is enquired after, and thence going back by Consequences to what is confessedly true.

Secondly, *Synthesis*, which is a Method of Enquiry or Demonstration in Mathematics, is, when we pursue the Truth chiefly by Reasons drawn from Principles before established, and *Propositions* formerly proved, and proceed by a long regular Chain, till we come to the Conclusion, as is done in the Elements of *Euclid*, and in almost all the Demonstrations of the Ancients: This is called Composition, and is opposed to the *Analytical* Method, which is called, Thirdly,

Resolution, a Method of Invention, whereby the Truth or Falshood of a Proposition, or its Possibility or Impossibility is discovered, in an Order contrary to that of *Synthesis* or Composition: For in this Analytical Method the Proposition is proposed as already known, granted or done; and then the Consequences thence deducible are examined, till at laſt we come to ſome known Truth or Falſhood, or Impoſſibility, whereof that which was propoſed is a neceſſary Conſequence, and from thence juſtly conclude the Truth or Impoſſibility of the Propoſition; which, if true, may then be demonſtrated in a *Synthetical* Method. This Method conſiſts more in the Judgment, Penetration and Readineſs of the Enquirer or Artiſt, than in any particular Rules; tho' thoſe of *Algebra* are of neceſſary Uſe, and a good Treaſure of *Geometry* in his Head will be of great Advantage to him in all Manner of Inveſtigations.

And therefore, " when any Problem or Queſtion (as " *Ward* has it) is propoſed to be Analytically reſolved; " it is very requiſite, that the true *Deſign* or *Meaning* " thereof be fully and clearly *comprehended* (in all its " Parts) that ſo it may be truly *abſtracted* from ſuch " *ambiguous Words*, as Queſtions of this kind are often " diſguiſed with, otherwiſe it will be very difficult, if " not impoſſible, to ſtate the Queſtion right in its ſub- " ſtituted Letters, and ever to bring it to an Equation," which the great and incomparable Mathematician, Sir *Iſaac Newton*, in his *Univerſalis*, ſays, that *Equations* are Ranks of Quantities, either equal to one another, or taken together, equal to Nothing. Theſe are to be conſidered chiefly after two Ways; either as the laſt Concluſions to which you come in the Reſolution of Problems; or as Means, by the Help whereof you are to obtain final Equations.

An Equation of the former kind is compoſed only out of one unknown Quantity involved with known ones, if the Problem be determined, and propoſes ſomething to be found out. But thoſe of the latter kind involve ſeveral unknown Quantities, which, for that Reaſon, muſt be compared among one another, and ſo connected, that out of all there may emerge a new Equation, in which there is only one unknown Quantity, which we ſeek, mixed with known Quantities. Which Quantity, that it may be the more eaſily diſcovered, that Equation muſt be transformed moſt commonly various Ways, untill it becomes the moſt ſimple that it can, and alſo like ſome of the following Degrees of them, in which x denotes the Quantity ſought, according to whoſe Dimenſions the

Terms,

Terms, as you ſee, are ordered, and p, q, r, s, t, &c. denote any other Quantities, from which, being known and determined, x is alſo determined, and may be inveſtigated by Methods in the enſuing Treatiſe hereafter to be explained.

$x=p$	Or, $x-p=0$
$xx=px+q$	$xx-px-q=0$
$x^3=px^2+qx+r$	$x^3-px^2-qx-r=0$
$x^4=px^3+qx^2+rx+s$	$x^4-px^3-qx^2-rx-s=0$
&c.	&c.

After this Manner therefore the Terms of Equations are to be ordered according to the Dimenſions of the unknown Quantity, ſo that thoſe may be in the firſt Place, in which the unknown Quantity is of the moſt Dimenſions, as x, xx, x^3, x^4, &c. and thoſe in the ſecond Place, in which x is of the next greateſt Dimenſions, as p, px, px^2, px^3, &c. As to what regards the Signs, they may ſtand any how; and one or more of the Intermediate Terms may be ſometimes wanting.

Thus $x^3 * -b^2x+b^3=0$, or $x^3=b^2x-b^3$, is an Equation of the third Degree, and $z^4+az^3-bz^3 * +ab^3-b^4=0$ is an Equation of the fourth Degree. For the Degree of an Equation is always eſtimated by the greateſt Dimenſion of the unknown Quantity, without any Regard to the known ones, or to the intermediate Terms. But by the Defect of the intermediate Terms, the Equation is moſt commonly render'd much more ſimple, and may be ſometimes depreſſed to a lower Degree. For thus $x^4=qxx+s$ is to be reckon'd an Equation of the ſecond Degree, becauſe it may be reſolved into two Equations of the ſecond Degree. For ſuppoſing $xx=y$, and y being accordingly writ for xx in that Equation, there will come out in its ſtead $yy=qy+s$, an Equation of the ſecond Degree, by the Help whereof, when y is found, the Equation $xx=y$ alſo of the ſecond Degree will give x.

And theſe are the Concluſions to which Problems are to be brought. But before I go upon their Reſolution by *Converging Series*, it will be neceſſary to ſhew the Method of transforming and reducing Equations into Order,

 and

and the Methods of finding the final Equations, according to Sir *Isaac Newton*, *viz.*

RULE I.

If there are any Quantities that may destroy one another, or may be joined into one by Addition or Subtraction, the Terms are that Way to be diminish'd.

EXAMPLE I.

Suppose =	1	$5b - 3a + 2x = 5a + 3x$
$1 + 3a$	2	$5b + 2x = 8a + 3x$
$2 - 2x$	3	$8a + x = 5b$
$3 - 8a$	4	$x = 5b - 8a.$

Again.

Suppose =	1	$\frac{2ab + bx}{a} - b = a + b$
$1 \times a$	2	$2ab + bx - ab = aa + ab$
$2 + ab$	3	$2ab + bx = aa + 2ab$
$3 - 2ab$	4	$bx = aa$
$4 \div b$	5	$x = \frac{aa}{b}.$

Q. E. D.

Again.

Suppose =	1	$abx + a^3 - a^2x = ab^2 - 2abx - x^3$
$1 + x^3$	2	$x^3 + abx + a^3 - a^2x = ab^2 - 2abx$
$2 + 2abx$	3	$x^3 + 3abx + a^3 - a^2x = ab^2$
$3 - ab^2$	4	$x^3 + 3abx + a^3 - ab^2 - a^2x = 0$ } they are both the same Equation.
Or,	5	$x^3 + 3abx - a^2x = ab^2 - a^3.$

Hence the same Equation may be solv'd by Converging Series, when we come to the other Part of this Treatise, where the Learner will meet with Plenty of Examples.

RULE II.

If there is any Quantity by which all the Terms of the Equation are multiplied, all of them must be divided by that Quantity, or if all are divided by the same Quantity, all must be multiplied by it too.

Suppose

Let $a=12$, $b=24$, $c=8$.

Suppose $=$	1	$\frac{b^3}{ac}-\frac{b^2x}{c^2}=\frac{x^2}{c}$
$1 \div c$	2	$\frac{b^3}{a}-\frac{b^2x}{c}=xx$
That is	3	$x^2-\frac{b^2}{c}x=-\frac{b^3}{a}$
3 in Numbers	4	$x^2-72x=-1152$
$4\ c\ \square$	5	$x^2-72x+1296=1296-1152=144$
$5\ \text{w}\ 2$	6	$x-36=\sqrt{144}=12$
$6+36$	7	$x=12+36=48.$ Q. E. D.

RULE III.

If there be any irreducible Fraction in whose Denominator there is found the Letter, according to whose Dimensions the Equation is to be ordered, all the Terms of the Equation must be multiplied by that Denominator, or by some Divisor of it.

Suppose	1	$\frac{ax}{a-x}+b=x$
$1 \times a-x$	2	$ax+ab-bx=ax-xx$
$2+xx$	3	$x^2+ax+ab-bx=ax$
$3-ax$	4	$x^2+ab-bx=0$
$4-ab$	5	$x^2-bx=-ab$ = a Quadratic of the 3d Form
$5\ c\ \square$	6	$x^2-bx+\frac{1}{4}b^2=\frac{1}{4}b^2-ab$
$6\ \text{w}\ 2$	7	$x-\frac{1}{2}b=\sqrt{\frac{1}{4}b^2-ab}$
$7\pm\frac{1}{2}b$	8	$x=\frac{1}{2}b\pm\sqrt{\frac{1}{4}b^2-ab}.$ Q. E. D.

RULE IV.

If that particular Letter, according to whose Dimensions the Equation is to be ordered, be involved with an irreducible Surd, all the other Terms are to be transposed to the other Side, their Signs being changed, and each Part of the Equation must be once multiplied by itself, if the Root be a Square one, or twice, if it be a Cubic one, &c.

Suppose

Suppose =	1	$\sqrt{aa-ax}+a=x$
1 − a	2	$\sqrt{aa-ax}=x-a$
2 ⊕ 2	3	$aa-ax=x^2-2ax+aa$
3 + ax	4	$aa=x^2-ax+aa$
4 − aa	5	$x^2-ax=0$
5 ÷ x	6	$x-a=0$
6 + a	7	$x=a$. Q. E. D.

Again.

Suppose =	1	$y=\sqrt{ay+y^2-a\sqrt{ay-y^2}}$
1 ⊕ 2	2	$yy=ay+y^2-a\sqrt{ay-y^2}$
2 ±	3	$ay=a\sqrt{ay-y^2}$
3 ÷ a	4	$y=\sqrt{ay-y^2}$
4 ⊕ 2	5	$yy=ay-y^2$
5 + y^2	6	$2y^2=ay$
6 ÷ $2y$	7	$y=\frac{a}{2}$. Q.E.D.

Again.

Suppose =	1	$\sqrt[3]{a^2x+2ax^2-x^3}-a+x=0$
1 + a − x	2	$\sqrt[3]{a^2x+2ax^2-x^3}=a-x$
2 ⊕ 3	3	$a^2x+2ax^2-x^3=a^3-3a^2x+3ax^2-x^3$
3 − a^2x	4	$2ax^2-x^3=a^3-4a^2x+3ax^2-x^3$
4 + x^3	5	$a^3-4a^2x+3ax^2=2ax^2$
5 + $4a^2x$	6	$a^3+3ax^2=4a^2x+2ax^2$
6 − a^3	7	$3ax^2=4a^2x+2ax^2-a^3$
7 − $2ax^2$	8	$ax^2=4a^2x-a^3$
8 ÷ a	9	$x^2=4ax-a^2$
9 + $4ax$	10	$x^2-4ax=-a^2=$ a Quadratic of the 3d Form.
10 c □	11	$x^2-4ax+4a^2=4a^2-a^2=3a^2$
11 w 2	12	$x-2a=\sqrt{3a^2}$
12 + $2a$	13	$x=2a\pm\sqrt{3a^2}$. Q. E. D.

RULE V.

The Terms, by Help of the preceding Rules, being disposed according to the Dimensions of some one of the Letters, if the highest Dimension of that Letter be multiplied

tiplied by any known Quantity, the whole Equation must be divided by that Quantity.

Suppose $=$	1	$2acx^3 - c^2x^3 + a^3x^2 + a^2cx^2 - 2ac^3x + a^2c^2x - a^3c^2 = 0$
$1 \div 2ac - c^2$	2	$x^3 + \frac{a^3x^2 + a^2cx^2 - 2ac^3x + a^2c^2x - a^3c^2}{2ac - cc}$
Or,	3	$x^3 + \frac{a^3 + a^2c}{2ac - cc}x^2 - a^2x - \frac{a^3c}{2a - c} = 0$
		Substitute $p = \frac{a^3 + a^2c}{2ac - c^2}$; $-a^2 = -q$, and $-\frac{a^3c}{2a - c} = -N$.
Equation is	4	$x^3 + px^2 - qx - N = 0$. Hence by Converging Series the Value of x may be found, and is shewn in the ensuing Treatise. Q. E. D.

RULE VI.

Sometimes Reduction may be performed by dividing the Equation by some compound Quantity.

N. B. *But I must beg leave to acquaint the Reader, that this Way is very difficult by inventing proper Divisors.*

Suppose $=$	1	$y^3 = by^2 - 2cy^2 + 3bcy - b^2c$
$1 \pm$	2	$y^3 + 2cy^2 - by^2 - 3bcy + b^2c = 0$
$2 \div y - b$	3	$y^2 + 2cy - bc = 0$
$3 + bc$	4	$y^2 + 2cy = bc$
$4c \square$	5	$y^2 + 2cy + c^2 = bc + c^2$
5 ω 2	6	$y + c = \overline{bc + c^2}\vert^{\frac{1}{2}}$
$6 - c$	7	$y = \overline{bc + c^2}\vert^{\frac{1}{2}} - c$. Q. E. D.

Here I shall give the Explanation of the 4th Step, *viz.*

$$
\begin{array}{l}
y - b)\; y^3 + 2cy^2 - by^2 - 3bcy + b^2c \;(y^2 + 2cy - bc. \\
\underline{y^3 - by^2} \\
\quad 2cy^2 - 3bcy \\
\quad \underline{2cy^2 - 2bcy} \\
\qquad\quad bcy + b^2c \\
\qquad\quad \underline{bcy + b^2c} \\
\qquad\quad 0 \quad 0
\end{array}
$$

RULE

RULE VII.

Sometimes also the Reduction is performed by Extraction of the Root out of each Part of the Equation.

Thus =	1	$xx = ax - b^2$
1 − ax	2	$x^2 - ax = -b^2$
2 c □	3	$x^2 - ax + \frac{1}{4}aa = \frac{aa}{4} - b^2$
3 ω 2	4	$x - \frac{1}{2}a = \sqrt{\frac{aa}{4} - b^2}$
4 ± $\frac{1}{2}a$	5	$x = \frac{1}{2}a \pm \sqrt{\frac{aa}{4} - b^2}$.

And thus universally, if you have $x^2 = px . q$; x will be $= \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp . q}$. where $\frac{1}{2}p$ and q are to be affected with the same Signs as p and q in the former Equation, but $\frac{1}{4}pp$ must always be made Affirmative.

For Example.

This =	1	$x^2 = px \pm q$
1 − px	2	$x^2 - px = \pm q$
2 c □	3	$x^2 - px + \frac{1}{4}pp = \frac{1}{4}pp . q$
3 ω 2	4	$x - \frac{1}{2}p = \sqrt{\frac{1}{4}pp . q}$
4 ± $\frac{1}{2}p$	5	$x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}pp . q}$. Q.E.D.

And this Example is a Rule, according to which all *Quadratic* Equations may be reduced to the Forms of Simple ones.

For Example.

Suppose $y^2 = \frac{2x^2y}{a} + x^2$. to extract the Root y, compare $\frac{2x^2}{a} = p$; and $x^2 = q$; that is, write $\frac{xx}{a}$ for $\frac{1}{2}p$, and $\frac{x^4}{aa} + x^2$ for $\frac{1}{4}p^2 . q$; and there will arise $y = \frac{x^2}{a} + \sqrt{\frac{x^4}{a} + x^2}$, or $y = \frac{xx}{a} - \sqrt{\frac{x^4}{a^2} + x^2}$. After the same Way, the Equation $yy = ay - 2cy + a^2 - c^2$, by by comparing $a-2c$ with p, and a^2-c^2 with q, will give $y=\frac{1}{2}a-c\pm\sqrt{\frac{5}{4}a^2-ac}$.

Moreover, the Biquadratic Equation $x^4=-a^2x^2+ab^3$, where odd Terms are wanting, by Help of this Rule becomes $xx=-\frac{1}{2}a^2\pm\sqrt{\frac{1}{4}a^4+ab^3}$, and again extracting the Root $x=\sqrt{-\frac{1}{2}a^2\pm\sqrt{\frac{1}{4}a^4+ab^3}}$. And so in others. Q. E. D.

Here follows a few Examples promiscuously set for clearing Equations.

EXAMPLE 1.

Suppose =	1	$3x^2+5x=232$
$1\div3$	2	$x^2+\frac{5}{3}x=\frac{232}{3}$
2 c □	3	$x^2+\frac{5}{3}x+\frac{25}{36}=\frac{232}{3}+\frac{25}{36}$
That is	4	$x^2+\frac{5}{3}x+\frac{25}{36}=\frac{2784+25}{36}$
4 ω 2	5	$x+\frac{5}{6}x=\pm\sqrt{\frac{2809}{36}}=\pm\frac{53}{6}$
hence	6	$x=\frac{-5\pm53}{6}=8$. or $-9\frac{2}{3}$.

Q. E. D.

EXAMPLE 2.

Suppose =	1	$7x^2+9x=1498$.
$1\div7$	2	$x^2+\frac{9}{7}x=\frac{1498}{7}$
2 c □	3	$x^2+\frac{9}{7}x+\frac{81}{196}=\frac{1498}{7}+\frac{81}{196}$
That is,	4	$x^2+\frac{9}{7}x+\frac{81}{196}=\frac{42025}{196}$
4 ω 2	5	$x+\frac{9}{14}=\pm\sqrt{\frac{42025}{196}}=\pm\frac{205}{14}$
Hence	6	$x=\frac{9\pm205}{14}=14$, or $-15\frac{2}{7}$.

PROOF.

Here let us prove, that $x = -15\frac{2}{7}$.

	1	$x = -15\frac{2}{7}$ or $-\frac{107}{7}$
Ergo	2	$x^2 = \frac{11449}{49}$
And	3	$7x^2 = \frac{11449}{7}$
And	4	$+9x = 9 \times -\frac{107}{7} = \frac{-963}{7}$
Consequently	5	$7x^2 + 9x = \frac{11449 - 963}{7} = \frac{10486}{7} = 1498.$

Q. E. D.

Example 3.

Suppose =	1	$9x^2 - x = 140$
1 ÷ 9	2	$x^2 - \frac{x}{9} = \frac{140}{9}$
2 c □	3	$x^2 - \frac{x}{9} + \frac{1}{324} = \frac{140}{9} + \frac{1}{324}$
that is	4	$x^2 - \frac{x}{9} + \frac{1}{324} = \frac{5040+1}{324}$
4 ɯ 2	5	$x - \frac{1}{18} = \frac{\pm 71}{18}$
hence	6	$x = \frac{1 \pm 71}{18} = 4$, or $-3\frac{8}{9}$.

Now we will prove, that $x = -3\frac{8}{9}$.

PROOF.

$x = -3\frac{8}{9}$, or $\frac{-35}{9}$. Ergo $x^2 = \frac{1225}{81}$, and $9x^2 = \frac{1225}{9}$,

$= \frac{1225}{9}$, and $-x = -1 \times \frac{-35}{9} = \frac{35}{9}$. Ergo $9x$ $-x = \frac{1225+35}{9} = \frac{1260}{9} = 140$.

EXAMPLE 4.

Suppose =	1	$\frac{45}{2x+3} + \frac{116}{4x+5} = 7$
$1 \times \overline{2x+3}$	2	$45 + \frac{232x+348}{4x+5} = 14x+21$
$2 \times \overline{4x+5}$	3	$180x+225+232x+348=56x^2+70x$ $+84x+105$
3, viz.	4	$412x+225+348=56x^2+154x+105$
4, viz.	5	$412x+573=56x^2+154x+105$
5 ±	6	$258x+468=56x^2$
6 ±	7	$56x^2-258x=468$
7 ÷ 56	8	$x^2 - \frac{258}{56}x = \frac{468}{56}$
8 c □	9	$x^2 - \frac{258}{56}x + \frac{66564}{12544} = \frac{468}{56} + \frac{66564}{12544}$.
That is	10	$x^2 - \frac{258}{56}x + \frac{66564}{12544} = \frac{104832+66564}{12544}$
	11	$x^2 - \frac{258}{56}x + \frac{66564}{12544} = \frac{171396}{12544}$
11 ω 2	12	$x - \frac{258}{112} = \pm \frac{414}{112}$
Hence	13	$x = \frac{+258+414}{112} = 6$, or $= -1\frac{11}{28}$.

PROOF, that x is $= -1\frac{11}{28}$.

$x = -1\frac{11}{28}$, or $= \frac{-39}{28}$. *Ergo* $2x = \frac{-39}{14}$, and $2x+3 = \frac{-39}{14} + \frac{42}{14}$, or $+\frac{3}{14}$. Therefore $\frac{45}{2x+3}$ $= 45$, divided by $\frac{3}{14}$, or (which is the same thing) =

45 ×

$45 \times \frac{14}{3}$ by Fractional Division $= \frac{630}{3} = 210$. Again $4x = \frac{-39}{7}$, and $4x + 5 = \frac{-39}{7} + \frac{35}{7} = \frac{-4}{7}$, and $\frac{116}{4x+5} = 116$, divided by $\frac{-4}{7} = 116 \times \frac{-7}{4} = -203$. Therefore $\frac{45}{2x+3} + \frac{116}{4x+5} = 210 - 203 = 7$.

Q. E. D.

Example 5.

Suppose =	1	$\frac{2by}{a} = \overline{\frac{a^2c^2}{4b^2y^2} + \frac{c^2}{4y^2}}\Big)^{\frac{1}{2}} = \sqrt{\frac{a^2c^2}{4b^2y^2} + \frac{c^2}{4y^2}}$
1 ⊙ 2	2	$\frac{4b^2y^2}{a^2} = \frac{a^2c^2}{4b^2y^2} + \frac{c^2}{4y^2}$
$2 \times a^2$	3	$4b^2y^2 = \frac{a^4c^2}{4b^2y^2} + \frac{a^2c^2}{4y^2}$
$3 \times 4b^2y^2$	4	$16b^4y^4 = a^4c^2 + \frac{4b^2y^2a^2c^2}{4y^2}$
$4 \times 4y^2$	5	$64b^4y^6 = 4a^4c^2y^2 + 4b^2y^2a^2c^2$
$5 \div y^2$	6	$64b^4y^4 = 4a^4c^2 + 4b^2a^2c^2$
$6 \div 4$	7	$16b^4y^4 = a^4c^2 + b^2a^2c^2$
$7 \div 16b^4$	8	$y^4 = \frac{a^4c^2 + b^2a^2c^2}{16b^4}$
8 ɯ 4	9	$y = \sqrt[4]{\frac{a^4c^2 + b^2a^2c^2}{16b^4}} = \overline{\frac{a^4c^2 + b^2a^2c^2}{16b^4}}\Big)^{\frac{1}{4}}$.

Q. E. D.

Example 6.

Suppose =	1	$\frac{b+x}{b}\sqrt{b^2 - x^2} = a - x$
1 ⊙ 2	2	$\frac{b^2 + 2bx + x^2}{b^2} \times \overline{b^2 - x^2} = a^2 - 2ax + x^2$
That is	3	$\frac{b^4 + 2b^3x - 2bx^3 - x^4}{b^2} = a^2 - 2ax + x^2$

$3 \times b^2$

$3\times b^2$	4	$b^4+2b^3x-2bx^3-x^4=a^2b^2-2ab^2x+b^2x^2$
or thus	5	$-x^4-2bx^3+2b^3x+b^4=a^2b^2-2ab^2x+b^2x^2$
5 ±	6	$-x^4-2bx^3-b^2x^2+2b^3x+2ab^2x=a^2b^2-b^4.$

Which is an Equation of the fourth Power, or of four Dimensions, and may be solved according to our Method, that we shall lay down for Converging Series.

Example 7.

Equation,	1	$.2b^2-a^2+2ax-x^2=a^2-2ax+x^2-4bx$
or	2	$x^2-2ax-4bx+a^2=-x^2+2ax+2b^2-a^2$
1,2 ±	3	$2x^2-4ax-4bx=2b^2-2a^2$
$3\div 2$	4	$x^2-2ax-2bx=b^2-a^2$
4 c □	5	$x^2-2ax-2bx+a^2+b^2=2a^2b+2bb$
5 ω 2	6	$x-a-b=\sqrt{2ab+2bb}$
6 ±	7	$x=a+b+\sqrt{2ab+2bb}.$ Q. E. D.

Example 8.

Suppose =	1	$ax-x^2=b\sqrt{a^2-2ax+2x^2}$
1 ⊙ 2	2	$x^4-2ax^3+a^2x^2=b^2a^2-2ab^2x+2b^2x^2$
2 ±	3	$x^4-2ax^3+a^2x^2-2b^2x^2+2ab^2x=b^2a^2$
		Add to both Sides $a^2b^2+b^4$
3 c □	4	$x^4-2ax^3+a^2x^2-2b^2x^2+2ab^2x+a^2b^2+b^4=a^2b^2+b^4$
4 ω 2	5	$x^2-ax-b^2=-b\sqrt{a^2+b^2}=\sqrt{a^2b^2+b^4}$
$5+b^2$	6	$x^2-ax=b^2-b\sqrt{a^2+b^2}$
6 c □	7	$x^2-ax+\frac{1}{4}aa=\frac{1}{4}aa+b^2-b\sqrt{a^2+b^2}$
7 ω 2	8	$x-\frac{1}{2}a=\sqrt{\frac{1}{4}a^2+b^2-b\sqrt{a^2+b^2}}$
$9\mp\frac{1}{2}a$	9	$x=\frac{1}{2}a\pm\sqrt{\frac{1}{4}a^2+b^2-b\sqrt{a^2+b^2}}$

Q. E. D.

EXAMPLE 9.

Equation	1	$2a-x=\sqrt{4ax-4xx}+\sqrt{4ax-3xx-aa}$
1 ⊚ 2	2	$4a^2-4ax+x^2=4ax-4x^2\times2\sqrt{4ax-4x^2}\times\sqrt{4ax-3x^2-a^2}+4ax-3x^2-a^2$
2 ±	3	$5a^2-12ax+8x^2=2\sqrt{4ax-4x^2}\times\sqrt{4ax-3x^2-a^2}$
3 ⊚ 2	4	$25a^4-120a^3x+224a^2x^2-192ax^3+64x^4=$ $4\times\overline{4ax-4x^2}\times\overline{4ax-3x^2-a^2}=80a^2x^2-112ax^3-16a^3x+48x^4$
4 ±	5	$16x^4-80ax^3+144a^2x^2-104a^3x+25a^4=0$
$5\div2x-a$	6	$8x^3-36ax^2+54a^2x-25a^3=0$, hence by Converging Series the Value of x will be found.

EXAMPLE 10.

Equation	1	$\overline{bb+2bx+x^2-c^2}\|^{\frac{1}{2}}+\overline{b^2-2bx+x^2-c^2}\|^{\frac{1}{2}}=a$
1 ±	2	$\overline{b^2-2bx+x^2-c^2}\|^{\frac{1}{2}}=a-\overline{b^2+2bx+x^2-c^2}\|^{\frac{1}{2}}$
2 ⊚ 2	3	$b^2-2bx+x^2-c^2=a^2-2a\sqrt{b^2+2bx+x^2-c^2}+b^2+2bx+x^2-c^2$
3 ±	4	$a^2+4bx=2a\sqrt{b^2+2bx+x^2-c^2}$
4 ⊚ 2	5	$a^4+8a^2bx+16b^2x^2=4a^2b^2+8a^2bx+4a^2x^2-4a^2c^2$
5 ± &c	6	$x^2=\dfrac{4a^2b^2-4a^2c^2-a^4}{16b^2-4a^2}$
6 ш 2	7	$x=\sqrt{\dfrac{4a^2b^2-4a^2c^2-a^4}{16b^2-4a^2}}$.

Q. E. D.

EXAMPLE

EXAMPLE 11.

Equation	1	$2\times\frac{y^2+2ay+a^2}{\sqrt{2y^2+2ay+a^2}}=3\times\frac{y^2}{\sqrt{2y^2+2ay+a^2}}$
		Seeing the Divisors are both alike they vanish.
Then	2	$2y^2+4ay+2a^2=3y^2$
2 ±	3	$y^2-4ay=2a^2$
3 c □	4	$y^2-4ay+4a^2=6a^2$
4 ω 2	5	$y=2a+\sqrt{6a^2}$.

EXAMPLE 12.

Equation	1	$\frac{b^2+c^2-m^2+2y\sqrt{c^2-b^2+y^2}}{\sqrt[2]{c^4+2c^2y^2+y^4}}=d$
Then	2	$b^2+c^2-m^2+2y\times\sqrt{c^2-b^2+y^2}=2d\times\overline{c^2+y^2}$
2 ±	3	$2y\times\sqrt{c^2-b^2+y^2}=2dc^2+2dy^2-b^2-c^2+m^2$
		Substitute $2dc^2-b^2-c^2+m^2=2g$, then the Equation
becomes	4	$2y\times\sqrt{c^2-b^2+y^2}=2g+2dy^2$
4 ÷ 2	5	$y\times\sqrt{c^2-b^2+y^2}=g+dy^2$
5 ⊕ 2	6	$yyc^2-y^2b^2+y^4=g^2+2gdy^2+d^2y^4$
6 ±	7	$d^2y^4-y^4+2gdy^2+b^2y^2-c^2y^2=gg$. By putting the Equation into Numbers, the Value of y will be found, by compleating the Square,

EXAMPLE 13.

Equation	1	$\frac{\frac{a^2x^2}{c^2}}{\frac{2ebx+2ebc}{dc}}+\frac{ebx+ebc}{2dc}=\frac{x^2-\frac{a^2x^2}{c^2}}{\frac{2b}{c}x}-\frac{b}{2c}x$

1 × dc

1 × dc	2	$\frac{\frac{da^2x^2}{c} - dce^2}{2ebx + 2ebc} + \frac{ebx + ebc}{2dc} = \frac{c^2x^2 - a^2x^2}{2bcx} - \frac{b}{2c}x$
2 × c	3	$\frac{da^2x^2 - dc^2e^2}{2ebcx + 2ebc^2} + \frac{ebx + ebc}{2dc} = \frac{c^2x - a^2x}{2bc} - \frac{b}{2c}x$
2c Van.	4	$\frac{da^2x^2 - dc^2e^2}{ebx + ebc} + \frac{ebx + ebc}{d} = \frac{c^2x - a^2x}{b} - \frac{b^2}{b}x$
		Transpose $\frac{eb}{d}x$, and substitute m, for $\frac{c^2 - a^2 - b^2}{b} - \frac{eb}{d}$
Then	5	$\frac{da^2x^2 - dc^2e^2}{ebx + ebc} + \frac{ebc}{d} = mx$
5×x+c	6	$\frac{da^2x^2 - dc^2e^2}{eb} + \frac{ebc}{d}x + \frac{ebc^2}{d} = mx^2 + mcx$
Reduced	7	$d^2a^2x^2 - d^2c^2e^2 + e^2b^2cx + e^2b^2c^2 = ebdmx^2 + ebdcmx.$
		Hence the Value of x may easily be found, by compleating the Square. Q. E. D.

Example 14.

Equation	1	$\sqrt{\frac{x^2 + 3a^2}{4}} - \sqrt{\frac{x^2 - 3a^2}{4}} = \sqrt{\frac{ax^2}{b}}$
1 ⊙ 2	2	$\frac{x^2 + 3a^2}{4} - 2\sqrt{\frac{x^2 + 3a^2}{4}} \times \sqrt{\frac{x^2 - 3a^2}{4}} + \frac{x^2 - 3a^2}{4} = \frac{ax^2}{b}$
2 ±	3	$\frac{x^2}{2} - \frac{ax^2}{b} = \sqrt{x^2 + 3a^2} \times \sqrt{\frac{x^2 - 3a^2}{4}}$

3 ⊙ 2	4	$\frac{x^4}{4} - \frac{ax^4}{b} + \frac{a^2x^4}{b^2} = x^2 + 3a^2 \times \frac{x^2-3a^2}{4} = \frac{x^4}{4} - \frac{9a^4}{4}$
4 ±	5	$\frac{a^2x^4}{b^2} - \frac{ax^4}{b} = -\frac{9a^4}{4}$
5 ×	6	$4a^2x^4 - 4abx^4 = -9a^4b^2$
6 ÷ a	7	$4bx^4 - 4ax^4 = 9a^3b^2$
7 ÷	8	$x^4 = \frac{9a^3b^2}{4b-4a}$
8 ω 4	9	$x = \sqrt[4]{\frac{9a^3b^2}{4b-4a}}$. Q. E. D.

Example 15.

Suppose	1	$\overline{xx - \frac{a^4}{x^2}}\Big\vert^{\frac{1}{2}} + \overline{aa - \frac{a^4}{x^2}}\Big\vert^{\frac{1}{2}} = \frac{xx}{a}$
1 ±	2	$\frac{xx}{a} - \overline{aa - \frac{a^4}{xx}}\Big\vert^{\frac{1}{2}} = \overline{xx - \frac{a^4}{x^2}}\Big\vert^{\frac{1}{2}}$
2 ⊙ 2	3	$\frac{x^4}{a^2} - \frac{2x^2}{a} \times \overline{aa - \frac{a^4}{xx}}\Big\vert^{\frac{1}{2}} + a^2 - \frac{a^4}{xx} = x^2 - \frac{a^4}{xx}$
3 ±	4	$\frac{x^4}{aa} + \frac{aa}{xx} = \frac{2x^2}{a} \times \overline{aa - \frac{a^4}{xx}}\Big\vert^{\frac{1}{2}}$
4 ⊙ 2	5	$\frac{x^8}{a^4} + 2x^4 - \frac{2x^6}{a^2} + a^4 - 2a^2x^2 + x^4 = 4x^4 - 4a^2x^2$
5 ±	6	$\frac{x^8}{a^4} - \frac{2x^6}{a^2} - x^4 + 2a^2x^2 + a^4 = 0.$
6 reduc'd	7	$x^8 - 2a^2x^6 - a^4x^4 + 2a^6x^2 + a^8$
7 ÷	8	$x^4 - aaxx - a^4 = 0.$
8 + a^4	9	$x^4 - a^2x^2 = a^4$
9 c □	10	$x^4 - a^2x^2 + \frac{1}{4}aaaa = a^4 + \frac{1}{4}a^4 = \frac{5}{4}a^4$
10 ω 2	11	$x^2 - \frac{1}{2}a^2 = \overline{\frac{5}{4}a^4}\Big\vert^{\frac{1}{2}}$

I 11 ±

11 ±	12	$x^2 = \frac{1}{2}aa + \overline{\frac{5}{4}a^4}\vert^{\frac{1}{2}}$
12 ω 2	13	$x = \overline{\frac{1}{2}aa + \overline{\frac{5}{4}a^4}\vert^{\frac{1}{2}}}\vert^{\frac{1}{2}}$ That is
$x = \sqrt{\frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}}$	14	$x = \sqrt{\frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}}$. Both which

are the fame, but a different Way of Expreffion, the latter is called the old, and the former new Method.

But left the Learner fhould be non plus'd how I divide the feventh Step, I thought proper to give him this

EXPLANATION.

$$x^4 - a^2x^2 - a^4)\,x^8 - 2a^2x^6 - a^4x^4 + 2a^6x^2 + a^8\,(x^4 - a^2x^2 - a^4$$

$$\begin{array}{l} x^8 - a^2x^6 - a^4x^4 \\ \hline -a^2x^6 + 2a^6x^2 \\ -a^2x^6 + a^4x^4 + a^6x^2 \\ \hline -a^4x^4 + a^6x^2 + a^8 \\ -a^4x^4 + a^6x^2 + a^8 \\ \hline 0 \quad 0 \quad 0 \end{array}$$

EXAMPLE 16.

Given	1	$d\sqrt{1 - ss} = \dfrac{sa}{\sqrt{1 - \dfrac{aa}{ss}}}$
1 ☉ 2	2	$dd \times \overline{1 - ss} = \dfrac{s^2a^2}{ss - aa}$
That is	3	$dd - ddss = \dfrac{ssaa}{ss - aa}$
3 ×	4	$d^2s^2 + d^2s^2a^2 - d^2s^4 - d^2a^2 = s^2a^2$
4 ÷ d^2	5	$s^2 + s^2a^2 - s^4 - a^2 = \dfrac{s^2a^2}{d^2}$
5 ±	6	$s^4 - s^2 - s^2a^2 + a^2 + \dfrac{a^2s^2}{d^2} = 0.$

EXAMPLE

EXAMPLE 17.

Given	1	$\overline{b^2+ax^2}\vert^{\frac{1}{2}} \times \overline{c^2+ax^2}\vert^{\frac{1}{2}} \times s = dx \times \overline{b^2-x^2}\vert^{\frac{1}{2}} + \overline{c^2-x^2}\vert^{\frac{1}{2}}$
1 ☉ 2	2	$\overline{b^2+ax^2 \times c^2+ax^2} \times s^2 = d^2x^2 \times \overline{b^2-x^2+c^2-x^2} + 2dx \times \overline{b^2-x^2}\vert^{\frac{1}{2}} \times \overline{c^2-x^2}\vert^{\frac{1}{2}}$
That is	3	$\frac{s^2 \times \overline{a^2x^4+b^2ax^2+c^2ax^2+b^2c^2}}{d^2x^2} + 2x^2 - b^2 - c^2 = 2dx \times \overline{x^4-b^2x^2-c^2x^2+b^2c^2}\vert^{\frac{1}{2}}$. Hence by involving both

Sides of the Equation, and ordering and tranſpoſing the Terms you will find the Value of x will ariſe to an adfected Equation of the eighth Power. Q. E. D.

Of Transforming *and* Exterminating *the* Unknown Quantities.

WHEN in the Solution of any Problem, there are more Equations than one to comprehend the State of the Queſtion, in each of which there are ſeveral unknown Quantities, thoſe Equations (two by two, if there are more than two) are to be ſo connected, that one of the unknown Quantities may be made to vaniſh at each of the Operations, and ſo produce a new Equation.

And to exterminate Quantities, you muſt obſerve.

Firſt, to know whether the propoſed Equations be really diſtinct, and depend on one another: And Secondly, whether they are more or fewer, or whether they be equal in Numbers with the unknown Quantities, which are contained in them; for, if you have never ſo many unknown Quantities, yet, if you have as many Equations for them, they may very eaſily by Reduction be brought to one Equation, where only one unknown Quantity ſhall

 remain,

remain, for all the unknown you may cast out at different Substitutions in your Operation, except one, which you may find in known Quantities, and then after this, or any one, is found, you may find the others by the Relation you will find them to have by your Operation.

But suppose you cannot find the several unknown Quantities in the several Equations, their Expressions may be discovered by a Process not much alike, I mean, by finding divers Ways by which the same unknown may be express'd in the Terms of the others unknown and given Quantities: Therefore the incomparable Sir *Isaac Newton* has given us the following Rules.

I. When the Quantity to be exterminated is only of one Dimension in both Equations, both its Values are to be sought by the foregoing Rules, and the one made equal to the other; as *Example* 1. 2. 3. &c.

II. When at least in one of the Equations the Quantity to be exterminated is only of one Dimension, its Value is to be sought in that Equation, and then to be substituted in its Room in the other Equation; as *Example* 5. &c.

III. When the Quantity to be exterminated is of more than one Dimension in both the Equations, the Value of its greatest Power must be sought in both, then if those Powers are not the same, the Equation that involves the lesser Power must be multiplied by the Quantity to be taken away, or by its Square, Cube, &c. that it may become of the same Power with the other Equation: Then the Values of those Powers are to be made equal, and there will come out a new Equation, where the greatest Power or Dimension of the Quantity to be taken is diminished; and by Repetition the Quantity will be exterminated.

EXAMPLE I.

Thus Suppose	1	$a+y=b+z$	Quere y, and how to exterminate z.
	2	$2y+z=3b$	
$1-b$	3	$a+y-b=z$	Now seeing we have the Value of z in both Steps substitute its Value.
$2 \pm$	4	$z=\frac{3b}{2y}$	
Thus	5	$a+y-b=\frac{3b}{2y}$	
$5 \pm$ &c.	6	$y=\frac{4b-a}{3}$.	Q. E. D.

EXAMPLE

EXAMPLE 2.

Suppose	1	$ax-2by=ab$ } Quere x. here y is to be exterminated.
	2	$xy=bb$
1 + 2	3	$ax-2by+xy=ab+b^2$
3 — ab	4	$ax-ab-2by+xy=b^2$
4 ± &c.	5	$\frac{ax-ab}{2b}=y=\frac{bb}{x}$
5 × x	6	$\frac{ax^2-abx}{2b}=b^2$
6 × 2b	7	$ax^2-abx=2b^3$, which reduced by c □
hence	8	$x=\frac{1}{2}b\pm\sqrt{\frac{2b^3}{a}+\frac{1}{4}b^2}$.

Q. E. D.

EXAMPLE 3.

Suppose	1	$x+y-z=0$ } exterminate z.
and	2	$ay=xz$
1 + z	3	$x+y=z$
2 ÷ x	4	$\frac{ay}{x}=z$
3=4	5	$\frac{ay}{x}=x+y$. Hence by Reduction we get
	6	$x^2+xy=ay$

Q. E. D.

EXAMPLE 4.

Suppose	1	$x^2+5x=3y^2$ } Quere x and y.
	2	$2xy-3x^2=4$
1 — 5x	3	$x^2=-5x+3y^2$
2 ±	4	$3x^2=2xy-4$
4 ÷ 3	5	$x^2=\frac{2xy-4}{3}$ { Seeing the 3d and 5th Steps are = Subst. its Value.
3 = 5	6	$\frac{2xy-4}{3}=-5x+3y^2$, reduced we get
This	7	$x=\frac{9y^2+4}{2y+15}$ { Now we have got the Value of x, we put its Value in the first Step, and the Equation is

per

per Sub.	8	$\frac{81y^4+72y^2+16}{4y^2+60y+225}+\frac{45y^2+20}{2y+15}=3y^2$
		Which reduced out of Fractions is
	9	$69y^4-90y^3+72y^2+40y+316=0.$

EXAMPLE 5.

Suppose	1	$xy^2=b^3$
and	2	$x^2+y^2=by-ax$
$1\div y^2$	3	$x=\frac{b^3}{y^2}$ substitute $\frac{b^3}{y^2}$ for x then the second Step.
becomes	4	$\frac{b^6}{y^4}+y^2=by+\frac{ab^3}{y^2}$, which reduced
is	5	$y^6-by^5+ab^3y^2+b^6=0.$

EXAMPLE 6.

Suppose	1	$x^3+y^3=a$ } required x, and y.
	2	$y^3-x^3=b$ }
$1-x^3$	3	$y^3=a-x^3$
$3+x^3$	4	$y^3=b+x^3$
$3=4$	5	$a-x^3=b+x^3$
$5+x^3$	6	$a=b+2x^3$
$6-b$	7	$2x^3=a-b$
$7\div 2$	8	$x^3=\frac{a-b}{2}$
8 *ω* 3	9	$x=\sqrt[3]{\frac{a-b}{2}}=\overline{\frac{a-b}{2}}\vert^{\frac{1}{3}}.$ Q. E. D.

UNIVERSALLY.

Let $m=$ the Power of the unknown Quantity.

Then {	1	$x^m-x^m=a$ } x and $y=$?
	2	$y^m-x^m=b$ }
$1-y^m$	3	$x^m=a-y^m$
$2-y^m$	4	$-x^m=b-y^m$
$\surd-x^m$	5	$-y^m=a-x^m$

$5+x^m$

$5 + x^m$	6	$-y^m = b + x^m$
$5 = 6$	7	$a - x^m = b + x^m$
$7 + x^m$	8	$a = b + 2x^m$
$8 - b$	9	$2x^m = a - b$
$9 \div \overline{2}$	10	$x^m = \frac{a-b}{2}$
10 ɯ 2	11	$x = \sqrt[m]{\frac{a-b}{2}} = \overline{\frac{a-b}{2}}^{m}$. Q. E. D.

Three unknown Quantities, how to find their Values.

EXAMPLE 7.

Suppose	1	$ax = zy$
	2	$x + z = \frac{y}{c}$
	3	$bx = z + y$
$1 \div a$	4	$x = \frac{zy}{a}$
$2 - z$	5	$x = \frac{y}{c} - z$
$4 = 5$	6	$\frac{zy}{a} = \frac{y}{c} - z$
$6 \times a$	7	$zy = \frac{ay}{c} - az$
$7 \times c$	8	$czy = ay - caz$
$8 + caz$	9	$czy + caz = ay$
$9 \div$	10	$z = \frac{ay}{cy + ca}$
5,	11	$x = \frac{y}{c} - z$
3,	12	$x = \frac{z+y}{b}$
$11 = 12$	13	$\frac{y}{c} - z = \frac{z+v}{b}$
13 reduced	14	$by - bcz = cz + cy$

14 ÷ ±

14 ÷ ±	15	$z = \frac{by - cy}{c + bc}$
10 = 15	16	$\frac{ay}{cy + ca} = \frac{by - cy}{c + bc}$
16 red.	17	$acy + abcy = bcy^2 - c^2y^2 + abcy - c^2ay$
17 − $abcy$	18	$acy = bcy^2 - c^2y^2 - c^2ay$
18 ÷ cy	19	$a = by - cy - ca$
20 + ca	20	$a + ca = by - cy$
20 ÷	21	$y = \frac{a + ac}{b - c}$ Q. E. D.

Example 8.

Suppose	1	$nx + my - pz = a$	Quere x, y, and z.
	2	$rx - qy + sz = b$	
	3	$gx + hy - lz = c$	

1 + pz	4	$nx + my = a + pz$
4 − my	5	$nx = a + pz - my$
5 ÷ n	6	$x = \frac{a + pz - my}{n}$
2 ±	7	$rx = b + qy - sz$
7 ÷ r	8	$x = \frac{b + qy - sz}{r}$
3 ±	9	$gx = c - hy - lz$
9 ÷ g	10	$x = \frac{c - hy - lz}{g}$
6=8=10	11	$\frac{a + pz - my}{n} = \frac{b + qy - sz}{r} = \frac{c - hy - lz}{g}$
6 = 8	12	$\frac{a + pz - my}{n} = \frac{b + qy - sz}{r}$
12 reduc'd	13	$ra + prz - mry = bn + qny - nsz$
13 ±	14	$prz + snz = bn + qny - ra + mry$
14 ÷	15	$z = \frac{bn + qny - ra + mry}{pr + sn}$
6 = 10	16	$\frac{a + pz - my}{n} = \frac{c - hy + lx}{g}$
16 reduc'd	17	$ga + gpz - gmy = cn - hny + lnz$

17 ±

17 ±	18	$lnz - gpz = ga - gmy - cn + nby$
18 ÷	19	$z = \frac{ga - gmy - cn + nby}{ln - gp}$
15 = 19	20	$\frac{bn + qny - ra + mry}{pr + sn} = \frac{ga - gmy - cn + nby}{ln - gp}$
20 red.	21	$bln^2 + qn^2ly - alnr + lmnry - bgpn - qgnpy + ragp - mgpry = ragp - prgmy - prcn + prnby + ansg - gmnsy - scn^2 + sbn^2y$
21 ±	22	$bln^2 - alnr - bgpn + prcn - ansg + scn^2 = prnby - gmnsy + sbn^2y - qn^2ly - lmnry + qgpy$
22 ÷	23	$y = \frac{bln^2 - alnr - bgpn + prcn - ansg + scn^2}{prbn - gmns + sbn^2 - qn^2l - lmnr + qgp}$. Q. E. D.

Example 9.

Suppose	1	$x + nz + ny + nu = a$	Quere x, y, z, u.
	2	$z + mx + my + mu = b$	
	3	$y + px + pz + pu = c$	
	4	$u + qx + qz + qy = d$	

1 ±	5	$x = a - nz - ny - nu$
2 ±	6	$mx = b - z - my - mu$
6 ÷ m	7	$x = \frac{b - z - my - mu}{m}$
5 = 7	8	$a - nz - ny - nu = \frac{b - z - my - mu}{m}$
8 reduced	9	$am - mnz - mny - mnu = b - z - my - mu$
9 ±	10	$am - mny - mnu - b + my + mu = mnz - z$
10 ÷	11	$z = \frac{am - mny - mnu - b + my + mu}{mn - 1}$
3 ±	12	$px = c - y - pz - pu$
12 ÷	13	$x = \frac{c - y - pz - pu}{p}$
7 = 13	14	$\frac{b - z - my - mu}{m} = \frac{c - y - pz - pu}{p}$
14 reduc'd	15	$pb - pz - pmy - pmu = mc - my - mpz - mpu$

15 ±	16	$mpz - pz = mc - pb + pmy - my$
16 ÷	17	$z = \frac{mc - pb + pmy - my}{mp - p}$
4, ±	18	$qx = d - u - qz - qy$
18 ÷ q	19	$x = \frac{d - u - qz - qy}{q}$
7 = 19	20	$\frac{b - z - my - mu}{m} = \frac{d - u - qz - qy}{q}$
20 reduc'd	21	$bq - qz - mqy - mqu = md - mu - mqz - mqy$
21 ±	22	$mqz - qz = md - mu - mqy - bq + qmu$
22 ÷	23	$z = \frac{md - mu - bq + qmu}{mq - q}$
1 = 19	24	$a - nz - ny - nu = \frac{d - u - qz - qy}{q}$
24 reduc'd	25	$qa - qnz - qny - qnu = d - u - qz - qy$
25 ±	26	$qa - qny - qnu - d + u + qy = qnz - qz$
26 ÷	27	$z = \frac{qa - qny - qnu - d + u + qy}{qn - q}$
23 = 27	28	$\frac{md - mu - bq + qmu}{mq - q} = \frac{qa - qny - qnu - d + u + qy}{qn - q}$
28 red.	29	$mqnd - mqnu - bq^2n + q^2mnu - qmd + qmu + bq^2 - q^2mu = mq^2a - mq^2ny - mq^2nu - mqd + mqu + mq^2y - q^2a + q^2ny + q^2nu + qd - qu - q^2y$
29 ± &c.	30	$y = \frac{mqnd - mqnu - bq^2n + bq^2 - mq^2a + 2mq^2nu + q^2a - q^2nu - qd + qu - q^2mu}{q^2m + q^2n - q^2 - mq^2n}$.

Q. E. D.

EXAMPLE

EXAMPLE 10.

Suppose	1	$xy^2+x^2y=z^3$
	2	$yz-xy=xz$
$2\pm$	3	$y=\frac{xz}{z-x}$. Put this for y in the first Step, then it will stand.
thus	4	$\frac{x^3z^2}{z^2-2zx+x^2}+\frac{x^3z}{z-x}=z^3$
$4\times$	5	$2x\,z^2-x^4z=z^5-2z^4x+z^3x^2$
$5\div z$	6	$2x^3z-x^4=z^4-2z^3x+z^2x^2$
$6\pm$	7	$z^4-2z^3x+z^2x^2-2x^3z+x^4=0$.

Q. E. D.

EXAMPLE 11.

Suppose	1	$x^2+x^2y^2z=a$
	2	$x^2y^2+x^2yz=b$
	3	$x^2y^2z^2+x^2y=c$
$1\pm$	4	$x^2=\frac{a}{1+y^2z}$
$2\pm$	5	$x^2=\frac{b}{y^2+yz}$
$3\pm$	6	$x^2=\frac{c}{y^2z^2+y}$
4=5=6	7	$\frac{a}{1+y^2z}=\frac{b}{y^2+yz}=\frac{c}{y^2z^2+y}$
7,	8	$\frac{a}{1+y^2z}=\frac{b}{y^2+yz}$
8 reduc'd	9	$ay^2+ayz=b+by^2z$
$9\pm$	10	$by^2z-ayz=ay^2-b$
$10\div$	11	$z=\frac{ay^2-b}{by^2-ay}$
5=6	12	$\frac{b}{y^2+yz}=\frac{c}{y^2z^2+y}$
$12\times$	13	$by^2z^2+by=cy^2+cyz$

13 ±	14	$by^2z^2 - cyz = cy^2 - by$
11 ⊙ 2	15	$\frac{a^2y^4 - b^2}{b^2y^4 - a^2y^2} = cy^2 - by$
Then	16	$z^2 = \frac{a^2y^4 - 2aby^2 + b^2}{b^2y^4 - 2aby^3 + a^2y^2}$
16 by^2	17	$by^2z^2 = \frac{a^2by^4 - 2ab^2y^2 + b^3}{b^2y^2 - 2aby + aa}$
11 × cy	18	$cyz = \frac{acy^2 - bc}{by - a}$
hence 14,	19	$\frac{a^2by^4 - 2ab^2y^2 + b^3}{b^2y^2 - 2aby + a^2} - \frac{acy^2 - bc}{by - c} = cy^2 - by.$
19. reduc'd	20	$a^2y^4 - bcy^4 + b^2y^3 + acy^3 - 4aby^2 + a^2y + bcy - ac + b^2 = 0.$

Hence an Equation of the 4th Power.

EXAMPLE 12.

Suppose	1	$\sqrt{\frac{ax^2}{px^2 - qx + n} - p} : - b = \sqrt{:x^2 - c:} - d.$
1 ±	2	$\sqrt{\frac{ax^2}{px^2 - qx + n} - p} = \sqrt{x^2 - c} + b - d$
		Put $m = b - d$
Then	3	$\sqrt{\frac{ax^2}{px^2 - qx + n} - p} = \sqrt{x^2 - c} + m$
3 ⊙ 2	4	$\frac{ax^2}{px^2 - qx + n} - p = x^2 - c^2 + 2m\sqrt{x^2 - c} + m^2$
4 ±	5	$\frac{ax^2}{px^2 - qx + n} - p - m^2 - c^2 - x^2 = 2m\sqrt{x^2 - c}$
		Now put $-b = -p - m^2 - c^2$
Then	6	$\frac{ax^2}{px^2 - qx + n} - b - x^2 = 2m\sqrt{x^2 - c}$
6 ⊙ 2	7	$\frac{a^2x^4}{p^2x^4 - 2pqx^3 + q^2x^2 + 2pnx^2 - 2qnx + n^2} - bb + 2bx^2 - x^4 = 4m^2 \times \overline{x^2 - c} = 4m^2x^2 - 4m^2c.$

Hence

Hence the seventh Step reduced out of Fractions, we get this final

Equation. $-p^2x^8+2pqx^7+2bp^2x^6-q^2x^6-2pnx^6+2qnx^5-4bpqx^5-p^2b^2x^4+2bq^2x^4+4pnbx^4-n^2x^4+2pqb^2x^3-4bqnx^3+2bn^2x^2-q^2b^2x^2+2pnb^2x^2+2qnb^2x-b^2n^2=4m^2p^2x^6-8m^2pqx^5+4m^2q^2x^4+8m^2pnx^4-4m^2cp^2x^4+8m^2cpqx^3-8m^2qnx^3+4m^2n^2x^2-4m^2cq^2x^2-8m^2cpnx^2+8m^2cqnx-4m^2n^2c.$

Hence the Co-efficients put into Numbers, and order'd, we shall have the Equation desired. Q. E. D.

EXAMPLE 13.

Suppose {	1	$xy^2=a^3$
	2	$x^2+y^2+bx=cy$
$1\div x$	3	$y^2=\frac{a^3}{x}$
3 ш 2	4	$y=\sqrt{\frac{a^3}{x}}$ Subst. its Value in the 2d.
2 Subt.	5	$x^2+\frac{a^3}{x}+bx=c\sqrt{\frac{a^3}{x}}$
$5\times x$	6	$x^3+a^3+bx^2=c\sqrt{a^3x}$
6 ⊛ 2	7	$x^6+2bx^5+b^2x^4+2a^3x^3+2a^3bx^2+a^6=c^2\times a^3x=c^2a^3x$
$7\pm$	8	$x^6+2bx^5+b^2x^4+2a^3x^3+2a^3bx^2-c^2a^3x+a^6=0.$

Hence by my Method of Converging Series, the Value of x may be found by putting the Co-efficients into Numbers.

Or the above Equation may be solv'd thus, *viz.*

Suppose {	1	$xy^2=a^3$
	2	$xx+yy+bx=cy$
$1\div y^2$	3	$x=\frac{a^3}{y^2}$ } Here put their Values in the Equation.
3 ⊛ 2	4	$x^2=\frac{a^6}{y^4}$ }

2,

2,	5	$\frac{a^6}{y^4} + y^2 + b\frac{a^3}{y^2} = cy$
$5 \times y^4$	6	$a^6 + y^6 + b\frac{a^3y^4}{y^2} = cy^5$
$6 \times y^2$	7	$a^6y^2 + y^8 + ba^3y^4 = cy^7$
$7 \div y^2$	8	$a^6 + y^6 + ba^3y^2 = cy^5$
$8 \pm$	9	$y^6 - cy^5 + ba^3y^2 + a^6 = 0.$

Note, At the seventh Step I have multiplied by y^2, and the next Step divided by y^2, so there is no need of such Multiplication and Division, by Reason it being Part of the Square of y^6, but this for the Instruction of the Learner.

EXAMPLE 14.

Suppose	1	$x^3y^2 + x^3z^2 = a$
	2	$x^3y + x^3z^2y = b$
	3	$x^3z^2 + x^3z^2y^2 = c$
$1 \div$	4	$x^3 = \frac{a}{y^2 + z^2}$ (Each are equal to one another.)
$2 \div$	5	$x^3 = \frac{b}{y + z^2y}$
$3 \div$	6	$x^3 = \frac{c}{z^2 + z^2y^2}$
$4 = 5$	7	$\frac{a}{y^2 + z^2} = \frac{b}{y + z^2y}$
7 reduc'd	8	$by^2 + bz^2 = ay + az^2y$
$4 = 6$	9	$\frac{a}{y^2 + z^2} = \frac{c}{z^2 + z^2y^2}$
9 reduc'd	10	$cy^2 + cz^2 = az^2 + az^2y^2$
$10 \div$	11	$z^2 = \frac{cy^2}{a + ay^2 - c}$
$5 = 6$	12	$\frac{b}{y + z^2y} = \frac{c}{z^2 + z^2y^2}$
reduc'd	13	$bz^2 + bz^2y^2 = cy + cz^2y$

13 ÷	14	$x^2 = \frac{cy}{b+by^2-cy}$
11 = 14	15	$\frac{cy^2}{a+ay^2-c} = \frac{cy}{b+by^2-cy}$
15 reduc'd	16	$acy+acy^3-c^2y = bcy^2+bcy^4-c^2y^3$
16 ÷ yc	17	$a+ay^2-c = by+by^3-cy^2$
17 ±	18	$by^3-cy^2-ay^2+by = a-c.$

Example 15.

Suppose	1	$x^2+yz=a$	Substitute
	2	$y^2+xz=b$	
	3	$z^2+xy=c$	$rx=y$; and $sx=z$.
Then 1	4	$x^2+rsx^2=a$	*Per* Substit. as above.
2	5	$r^2x^2+sx^2=b$	
3	6	$s^2x^2+rx^2=c$	
Then 4,	7	$x^2=\frac{a}{1+rs}$	
5	8	$x^2=\frac{b}{r^2+s}$	
6	9	$x^2=\frac{c}{s^2+r}$	
7=8=9	10	$\frac{a}{1+rs}=\frac{b}{r^2+s}=\frac{c}{s^2+r}$	
10, red.	11	$ar^2+as=b+brs$	
7, 9,	12	$as^2+ar=c+crs$	
11 ±	13	$as-brs=b-ar^2$	
13 ÷	14	$s=\frac{b-ar^2}{a-br}$	
14 ⊙ 2	15	$s^2=\frac{b^2-2abr^2+a^2r^4}{a^2-2abr+b^2r^2}=\frac{\overline{b-ar^2}\vert^2}{\overline{a-br}\vert^2}$	
		Now substitute for the Value of s in the 12th Step, and it	
is	16	$a\times\frac{\overline{b-arr}\vert^2}{\overline{a-br}\vert^2}+ar=c-cr\times\frac{b-ar^2}{a-br}$	

That

That is	17	$a\times\overline{b-arr}^2+ar\times\overline{a-br}^2=c\times\overline{a-br}^2 - cr\times b-arr\times a-br$
That is	18	$ab^2-2abr^2+a^3r^4+a^3r-2a^2br^4+ab^2r^3 = ca^2-2abcr+b^2cr^2-abcr+a^2cr^3+b^2cr^2-abcr^4$
18 ±	19	$a^3r^4+abcr^4+ab^2r^3-a^2cr^3-2b^2cr^2-2a^2br^2-2abr^2+a^3r+3abcr=ca-ab^2$.

Lest the Learner ſhould not be oble to make out the 16, or 17, Steps, I thought fit to ſet it down here as an *Explanation*, viz.

$$16\quad \frac{ab^2-2a^2br^2+a^3r^4}{a^2-2abr+b^2r^2}+ar=c+\frac{bcr-abcr^3}{a-br}.$$

Which, if this laſt reduc'd out of Fractions according to the common Rules of Algebra, will give the Equation as the 18th Step exhibits.

EXAMPLE 16.

Suppoſe

$a\times \overline{ee+uu}=b$, $e\times\overline{aa+uu}=c$, $u\times\overline{ee+aa}=d$ Or $aee+auu=b$. $eaa+euu=e$. $uee+uaa=d$.

Quere the Value of a, e, u?

SUBSTITUTE

$x=u$		$xx=uu$.
$yx=e$	Then	$y^2x^2=e^2$
$zx=a$		$z^2x^2=a^2$.

Then the above Equations will ſtand, *viz.*

Equat:	1	$zx\times\overline{y^2x^2+x^2}=b=y^2x^3z+x^3z$
And	2	$yx\times\overline{z^2x^2+x^2}=c=x^3z^2y+x^3y$
And	3	$x\times\overline{y^2x^2+z^2x^2}=d=y^2x^3+z^2x^3$
1 ÷	4	$x^3=\dfrac{b}{y^2z+z}$

2 ÷	5	$x^3 = \frac{c}{yzz+y}$
3 ÷	6	$x^3 = \frac{d}{yy+zz}$
4=5=6.	7	$\frac{b}{zyy+z} = \frac{c}{yzz+y} = \frac{d}{yy+zz}$
7 ×	8	$byz^2 + by = czy^2 + cz$
viz.	9	$by^2 + bz^2y = dzy^2 + dz$
and	10	$cy^2 + cz^2 = dyz^2 + dy$
10 ±	11	$cy^2 - dy = dyz^2 - cz^2$
11 ÷	12	$z^2 = \frac{cy^2 - dy}{dy-c} = \frac{y \times \overline{cy-d}}{dy-c}$
		For z^2 put its Value in the above Equations
then 7	13	$z \times \overline{y^2+1} = \frac{by+y^2-1}{dy-c}$. Now this 13th Step divided by y^2+1, and squared it will be
13 ÷	14	$\frac{b^2y^2 \times \overline{y^2-1}\rvert^2}{\overline{dy-c} \times \overline{y^2+1}\rvert^2} = zz = \frac{\overline{cy-d} \times y}{dy-c}$
14 red.	15	$y^6 - \frac{dd+cc+bb}{dc}y^5 + 3y^4 - \frac{2d^2+2c^2+b^2}{dc}y^3 - \frac{dd+cc+bb}{dc}y + 1 = 0$. Hence y being found, we have then $z = \sqrt{\frac{cy^2-dy}{dy-c}}$ and $x = \overline{\frac{dy-c}{y \times y^2-1}}\rvert^{\frac{1}{2}}$.
		Q. E. J.

Example 17.

Suppose $a^3e^3 + ae = b$
and $a^3 + a^2e + ae^2 + e^3 = c$ } Quere a, and e.

Substitute $x = a+e$, and $z = ae$, then

	1	$x^3 = a^3 + 3a^2e + 3ae^2 + e^3$
	2	$2zx = 2a^2e + 2ae^2$
1−2	3	$x^3 - 2zx = a^3 + a^2e + ae^2 + e^3 = c$; and

 by

by substituting for its Value z, in the first $z^3 + z = b$. Now finding the Value of z, and then substituting its Value in the second Step, we shall have the Value of x.

Or, thus,

Substitute $xy = a$, and $\frac{x}{y}$ for e. in the two given Equations, then you will have $x^6 + x^2 = b$ for the first, where you will by our Method of Converging Series hereafter laid down find the Value of x. In the other Equation you have (first multiplying by y^3, and dividing by x^3) $y^6 + y^4 + y^2 + 1 = \frac{c}{x^3} y^3$. Consequently you will have the Value of y.

EXAMPLE 18.

Suppose $\overline{2z^2}|^{\frac{1}{2}} + \overline{3z^3}|^{\frac{1}{3}} + \overline{6z^6}|^{\frac{1}{6}} = a$.

The same Equation may be set thus,

$\overline{2}|^{\frac{1}{2}} \times z + \overline{3}|^{\frac{1}{3}} \times z + \overline{6}|^{\frac{1}{6}} \times z = a$.

Then $z = \frac{a}{\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{6}} =$ Answer.

Q. E. J.

Before I conclude this Part, I shall give a few Examples how to manage Surds, let them be never so much complicated, the young Reader may by what is here done be able to solve any Equation, especially, if he will observe the following, as Prefatory

RULES.

The Characters used by Algebraists are in most Authors shall not insert them here. Where Note, that the Square, Cube, Biquadratic, &c. of $x+y$ is $x^2+2xy+y^2$. $x^3+3x^2y+3xy^2+y^3$. $x^4+4x^3y+6x^2y^2+4xy^3+y^4$, &c. but there is a new Way for Notation in Operations as are used by the most acute Mathematicians; as

$\overline{x+y}|^2 =$ Square of $x+y = x^2+2xy+yy$.
$\overline{x+y}|^3 =$ Cube of $x+y = x^3+3x^2y+3xy^2+y^3$

$\overline{x+y}|$

$\overline{x+y}|^4$ = Biquadrate of $x+y = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

$\overline{x+y}|^5$ = Fifth Power of $x+y$, &c. &c.

Consequently the Square Root of

y^3+aay is $\sqrt{y^3+aay}$, or by the new Way $= \overline{y^3+a^2y}|^{\frac{1}{2}}$

Cube Root of $y^3+a^2y = \sqrt[3]{y^3+a^2y}$, or $\overline{y^3+a^2y}|^{\frac{1}{3}}$

Cube Root of the Square of $y^3+a^2y = \sqrt[3]{y^6+2a^2y^4+a^4y^2}$

$= \overline{\overline{y^3+a^2y}|^2}|^{\frac{2}{3}} = \sqrt[3]{\overline{y^3+a^2y}|^2}$.

&c. &c.

UNIVERSALLY.

$\sqrt[3]{a^2b-c^3+d^3}$, or $\overline{a^2b-c^3+d^3}|^{\frac{1}{3}}$, is the Cube Root of the Difference between a^2b+d^3 and c^3.

$\sqrt[3]{a^3\pm\sqrt{ab+g+nx}}$, or $\overline{a^3+\overline{ab+g+nx}|^{\frac{1}{2}}}|^{\frac{1}{3}}$, is the Cube Root of a^3 added or subtracted to or from the Square Root of $ab+g+nx$.

And $\sqrt[4]{m^4+\sqrt[3]{a^3p^5+b^3c^3-dd}\sqrt{aa+cc}}$, or $=$ $\overline{m^4}|^{\frac{1}{4}} + \overline{a^3p^5+b^3c^3-dd\times\overline{aa+cc}|^{\frac{1}{2}}}|^{\frac{1}{3}}$, is the Biquadrate Root from m^4 added to the Cube Root $a^3p^5+b^3c^3-dd$, multiplied into the Square Root of $aa+cc$. And from these the young Algebraist will be able to determine the rest.

But Surds may be reduced to facilitate Operations, which to do please to observe this

RULE.

Divide the Surd by the greatest Square, Cube, Biquadrate, &c. or any other higher Power, by which you can discover, is contained in it, and it will measure it without any Remainder, and then prefix the Root of that Power before the Quotient, or Surd so divided, and this will produce a new Surd of the same Value with the former, but in more simple Terms; as

$\overline{49abb}|^{\frac{1}{2}}$
$49bb$

$\frac{\overline{49abb}|^{\frac{1}{2}}}{49bb}$; the Root of which Quotient ($49b^2$) is $7b$, which by the Rule must be prefix'd thus, $7b\sqrt{a}$, or $7b \times \overline{a}|^{\frac{1}{2}}$. Also $\sqrt[3]{xy^3z} = \overline{xy^3z}|^{\frac{1}{3}}$ will, by the same Way of Reasoning be $y\sqrt[3]{zx} = y \times \overline{xz}|^{\frac{1}{3}}$. And this Reduction is of great Use; and besides it looks neater and workman-like to express Quantities in their most simple Terms.

Suppose you would Square, Cube, &c. any Surd Root, it is no more than to Square, Cube, &c. the Power retaining the same Note of Radicality; as for Instance. Suppose you would Cube $\sqrt[6]{aab} = \overline{aab}|^{\frac{1}{6}}$, it will be $\sqrt{aab} = \overline{aab}|^{\frac{1}{2}}$. Again, suppose $\sqrt[9]{mx^3y^2} = \overline{mx^3y^2}|^{\frac{1}{9}}$ were to be Cub'd, it will be $\sqrt[3]{mx^3y^2} = \overline{mx^3y^2}|^{\frac{1}{3}}$. So the Biquadrate Root of $\sqrt{xy} = \overline{xy}|^{\frac{1}{2}}$ is x^2y^2, as being the Square of the Square of $\overline{xy}|^{\frac{1}{2}}$, and the Cube of $\sqrt[6]{xx} = \overline{xx}|^{\frac{1}{6}}$ will be $\overline{xx}|^{\frac{1}{2}}$, or x.

But it is better, where it can be done, to take one half, one third Part, &c. of the Exponent of the Root; on the contrary, if you would extract the Square, Cube, &c. Root of any Surd, you must double, triple, &c. the Exponents of the Radicality, thus the Square Root of $\sqrt{5mpx} = \overline{5mpx}|^{\frac{1}{2}}$ is

$\sqrt[4]{5mpx} = \overline{5mpx}|^{\frac{1}{4}}$, and the Square Root of

$\sqrt[3]{axy} = \overline{axy}|^{\frac{1}{3}}$ is $\sqrt[6]{axy} = \overline{axy}|^{\frac{1}{6}}$.

It will not be amiss to shew how $\frac{1}{x}$ may be multiplied by $\frac{1}{\sqrt[3]{x5}}$, or $\frac{1}{\overline{x5}|^{\frac{1}{3}}}$. Thus,

$$\frac{1}{x} \times \frac{1}{\sqrt[3]{x5}} = x^{-1} \times x^{\frac{5}{3}} = -\frac{3}{3} \times x^{\frac{5}{3}} = x - \frac{8}{3}$$

$$= \frac{-1}{x^{\frac{8}{3}}} = \frac{1}{\sqrt[3]{x^8}}.$$ But to note by the Way this Expression x^{-1}.

If

If a Series of Geometrical Progressionals be in this Order, 1. x. x^2. x^3. x^4. x^5. x^6. x^7. &c. Their Indexes or Exponents will be in Arithmetical Progression, and stand thus; 0. 1. 2. 3. 4. 5. 6. 7. &c. But if they are Fractions, as $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{1}{x^3}$, $\frac{1}{x^4}$, $\frac{1}{x^5}$, $\frac{1}{x^6}$, &c. then their Exponents will be Negative, and stand thus; —1. —2. —3. —4. &c, for if you suppose $x=2$, then will $\frac{1}{x}=\frac{1}{2}$, and $\frac{1}{x^2}=\frac{1}{4}$, and $\frac{1}{x^3}=\frac{1}{8}$, &c. Or if you expreſs the Geometrical Series by Means of the Exponents, it will stand x^{-1}, x^{-2}, x^{-3}, x^{-4}, x^{-5}, &c, Thus is $\frac{1}{x^4}=x^{-4}$, and $\frac{1^3}{x^3}=x^{-3}$, &c. And $1=x^0$. $x^1=x$. $x^2=xx$. &c. Alſo the Exponent of $\sqrt{x}$ will be $\frac{1}{2}$, becauſe as $\sqrt{x}$ is a mean Proportional between 1 and x, ſo $\frac{1}{2}$ is an Arithmetical Mean between 0 and 1.

And the Exponent of $\sqrt[3]{x}$ will be $\frac{1}{3}$, becauſe as $\sqrt[3]{x}$ is the firſt of the two mean Proportionals between 1 and x, ſo $\frac{1}{3}$ is the firſt of the two Arithmetical Means between 0 and 1.

Hence $\sqrt{x}$ and $\overline{x}|^{\frac{1}{2}}$, or $\sqrt[3]{x}$ and $\overline{x}|^{\frac{1}{3}}$, or $\sqrt[5]{x^4}$ and $\overline{x}|^{\frac{4}{5}}$ are the ſame, but two different Ways of Notation for one and the ſame Thing, the former being the old, the latter the new Method of Notation, as is ſpecified above.

1. Moreover, if any Rational Quantity be to be divided by its Square Root, the Square Root will be the Quotient; as for Inſtance, ſuppoſe mx be divided by $\sqrt{mx}$, the Quotient muſt be $\sqrt{mx}$.

2, When a Surd Root having a Rational Quantity prefix'd before it, is to be divided by the Surd Part of it, the Quotient will be the Rational Quantity. Thus $a\sqrt{mx}$ to be divided by $\sqrt{mx}$, the Quotient muſt be a, for

$$\frac{a\sqrt{mx}}{\sqrt{mx}}=a.$$

3. When the Dividend and Divisor are the Products of two Rational Quantities multiplied severally into one common Surd, or when they are Rational Quantities prefix'd before one common Surd; then divide the Rational Part of the Dividend by the Rational Part of the Divisor, and what results is the Quotient.

Thus $\frac{12\sqrt{mx}}{3\sqrt{mx}} = 4.$ Quotient. and $\frac{a^2b^2\sqrt{mx}}{ab\sqrt{mx}} = ab.$

4. But when the Dividend and Divisor are two Rational Quantities or Numbers prefix'd to two unequal Surds, then you must divide not only as before the Rational Part of the Dividend by that of the Divisor, but also the Surd Part, and these two Quotients connected together, so as the Rational Part should stand on the left Hand, are the true Quotient sought.

Thus $\frac{8\sqrt{30}}{4\sqrt{10}} = 2\sqrt{3,}$ and $\frac{mcx\sqrt{pxx}}{mx\sqrt{x}} = c\sqrt{px,}$ and

$\frac{cx\sqrt{aa}}{gd} = \frac{cx}{gd}\sqrt{a,}$ the Quotient.

Hence it may not be amiss to observe how $\frac{1}{\sqrt[3]{x5}}$ may be divided by $\frac{1}{x}$, which is thus.

$$\frac{1}{x}\Big)\frac{1}{\sqrt[3]{x5}}\Big(=x^{-1}\Big)\ x^{-\frac{5}{3}}\Big(=x^{-\frac{3}{3}}\Big)x^{-\frac{5}{3}}=$$

$$\Big(-\frac{2}{3}=\frac{1}{\sqrt[3]{xx}},\ \&c.$$

$$\text{Also } \frac{\sqrt{\frac{aaxx}{b^2}+\frac{4mnxx}{b^2}}}{\sqrt{aa+4mn}} = \frac{x}{b}\ \text{Quotient.}$$

$$\text{And } \frac{\sqrt{\frac{aammnn}{qqxx}+\frac{4bn^3}{qxx}}}{\sqrt{mm+4mn}} = \frac{bn}{qx}\ \text{Quotient.}$$

And

And $\dfrac{\sqrt{\dfrac{9a^2b - 12abx + 4bx^2 - 9a^2x + 12ax^2 - 4x^3}{bb}}}{\sqrt{b-x}} =$

$\sqrt{\dfrac{9a^2 - 12ax + 4x^2}{bb}} = \dfrac{3a-2x}{b}$ the Quotient.

Also $\dfrac{\sqrt{\dfrac{16bc^2 - 24bcx + 9bx^2 - 16c^2x + 24cx^2 - 9x^3}{dd}}}{\sqrt{b-x}} =$

$\sqrt{\dfrac{16c^2 - 24cx + 9x^2}{dd}} = \dfrac{4c-3x}{d}$ the Quotient.

I think it will be needleſs to add any more on this Score, but proceed to Equations, wherein Surds are concerned, and I muſt obſerve to the Reader that not to haſten too faſt before he is Maſter of what he is about, and in Time he will find the moſt arduous will be obvious.

EXAMPLE

Suppoſe $\sqrt{a+x-b} - \sqrt{c} = 2\sqrt{x} - \sqrt{x-b}$.

Then.

Equation	1	$\sqrt{a+x-b} - \sqrt{c} = 2\sqrt{x} - \sqrt{x-b}$
1 ⊙ 2	2	$a+c-2\sqrt{ac+cx-bc} = 4x - 4\sqrt{xx-bx}$
		For $a+c$ put m, and $a-b$ put n.
Then	3	$m - 2\sqrt{ac+cx-bc} = 4x - 4\sqrt{xx-bx}$
3 ±	4	$4x - m + 2\sqrt{cn+cx} = 4\sqrt{xx-bx}$
4 ⊙	5	$m^2 - 8mx + 16x^2 + 4cn + 4cx + \overline{16x-4m}\sqrt{cn+cx} = 16x^2 - 16bx$
5 + $16x^2$	6	$m^2 - 8mx + 4cn + 4cx + \overline{16x-4m}\sqrt{cn+cx} = -16bx$
		Now put p for $8m - 16b - 4c$, and q for $m^2 + 4cn$.

Then

Then ±	7	$\overline{16x-4m}\sqrt{\overline{cn+cx}}=px-q$
7 ⊙ 2	8	$256cnx^2+256cx^3-128cmnx-128cmx^2+16cm^2n+16cm^2x=p^2x^2-2pqx+q^2$.

Q. E. J.

EXAMPLE

Suppose $\sqrt{\overline{b+x}}-\sqrt{b}=\sqrt{\overline{a+c+x}}-\sqrt{a}$.

Then.

Equation	1	$\sqrt{\overline{b+x}},-\overline{b}|^{\frac{1}{2}}=\overline{a+c+x}|^{\frac{1}{2}}-\sqrt{a}$
1 ±	2	$\sqrt{\overline{b+x}}=\overline{a+c+x}|^{\frac{1}{2}}-\sqrt{a}+\sqrt{b}$ For $\sqrt{a}+\sqrt{b}$ put $\sqrt{m}$, and for $a+c$ put n
then	3	$\sqrt{\overline{b+x}}=\sqrt{\overline{n+x}}-\sqrt{m}$
3 ⊙ 2	4	$b+x=n+x+m-2\sqrt{\overline{mn+mx}}$
4 ±	5	$n+m-b=2\sqrt{\overline{mn+mx}}$, for $n+m-b$ put q.
then	6	$q^2=4mn+4mx$.

EXAMPLE

Suppose $\sqrt{x}+\frac{a^3}{c^3}\sqrt{x}=\overline{\frac{a^6}{c^6}x+p-x}|^{\frac{1}{2}}$.

Now for $\frac{a^3+c^3}{c^3}$ put $\frac{r}{s}$, and for $\frac{a^6-c^6}{c^6}$ put $\frac{rt}{ss}$, and

hence the Equation is	1	$\frac{r}{s}\sqrt{x}=\sqrt{\frac{rt}{ss}x+p}$
1 ⊙ 2	2	$\frac{rr}{ss}x=\frac{rt}{ss}x+p$
2 × and ±	3	$s^2r^2x=s^2rtx=s^4p$
3 ÷ s^2 and ±	4	$r^2x-rtx=s^2p$
4 ÷	5	$x=\frac{ssp}{rr-rt}$.

Here

Here you see how in Equations, where Substitution can be had, how much it conduces to the facilitating the Operations, and bring them out to Simple Equations; when at the same Time, if order'd in their Surds, would rise to Equations of the third, fourth, fifth, &c. Power. Perhaps the Reader will not understand at first Sight what is meant by $\frac{a^6 - c^6}{c^6}$ for $\frac{a^6 + c^6}{c^6}$, but if he will regard the Surd, he will find $-x$, from whence the above Substitution proceeds.

EXAMPLE.

Suppose the Reader should, in solving a Mathematical or Geometrical Problem have this Proportion.

$$ax : b\sqrt{b^2 + 2bx + x^2 - a^2} :: \sqrt{s - \frac{b^2}{c^2a^2} \times \overline{c^4 - x^2}} : \frac{b}{ca}\sqrt{c^2 - x^2}.$$

Now Euclid in his 6th Book, Prop. 16. says, when four Numbers are proportional, the Rectangle comprehended under the Extremes is equal to the Rectangle comprehended under the Means, Hence this

Equation, $\frac{bx}{c}\sqrt{c^2 - x^2} = b\sqrt{b^2 + 2bx + x^2 - a^2} \times \sqrt{s - \frac{b^2}{c^2a^2} \times \overline{c^2 - x^2}}$. But the same may be order'd without Surds thus, involve every Term in the Proportion, and then it will be

$$\text{As } a^2x^2 : b^2 \times \overline{b^2 + 2bx + x^2 - a^2} :: s - \frac{b^2}{c^2a^2} \times \overline{cc - xx} : \frac{b^2}{c^2a^2} \times cc - xx.$$

Now by multiplying Extremes and Means together, and due Reduction and Transposition, we have this

Equation. $\dfrac{\left.\begin{matrix} x^4 + 2b^3x^3 - a^2b^2 + 2a^2bc^2 + 2a^2b^2c^2 \\ + b^4 \; x^2 \qquad x - a^4c^2 \\ - b^2c^2 \quad - 2b^3c^2 \quad - b^4c^2 \end{matrix}\right\} = 0}{aa + bb}$

Hence this laſt Method is the eaſieſt and readieſt for Operation, for Squaring each Term in the Proportion, we immediately extenuate all the Radical Signs, which, when ſo involv'd, it is eaſy to bring it to an Equation by the 16th, E. 6. Q. E. I.

EXAMPLE.

Suppoſe	1	$bcx = bx - mx + \sqrt{n^2 - n^2x^2} + \sqrt{d^2 - d^2x^2}$
		For $bc + m - b$, put p. Then
Then 1	2	$px = \sqrt{n^2 - n^2x^2} + \sqrt{d^2 - d^2x^2}$
2 ☉ 2	3	$p^2x^2 = n^2 - n^2x^2 + d^2 - d^2x^2 + 2\sqrt{d^2n^2 - 2d^2n^2x^2 + d^2n^2x^4}$
		Again put $-q = -n^2 - d^2$; $s = p^2 + n^2 + d^2$
Then	4	$sx^2 - q = 2\sqrt{d^2n^2 - 2d^2n^2x^2 + d^2n^2x^4}$
4 ☉ 2	5	$s^2x^4 - 2qsx^2 + q^2 = 4d^2n^2 - 8d^2n^2x^2 + 4d^2n^2x^4$
5 ſub. &c.	6	$x = \sqrt{g + \sqrt{gg - h}}$

Q. E. I.

Here you may ſee how I ſubſtituted firſt for p, and $-q$, and s, So by this Subſtitution we have a Quadratic Equation.

EXAMPLE.

Suppoſe $\dfrac{\sqrt{x^2 - c^2}}{x^2} : \dfrac{2cd - 2cy}{\overline{y^2 - c^2}|^{\frac{1}{2}}} :: \dfrac{\overline{y^2 - c^2}|^{\frac{1}{2}}}{y^2} : \dfrac{2cb - 2cx}{\overline{x^2 - y^2}|^{\frac{1}{2}}}$

: Then *per* E. 6. 16. by multiplying Extremes and Means, we get this

Equation

Equation	1	$\frac{\overline{2cb-2cx}\,\sqrt{x^2-c^2}}{x^2\sqrt{x^2-c^2}} = \frac{\overline{2cd-2cy}\,\sqrt{y^2-c^2}}{y^2\sqrt{y^2-c^2}}$
1 ÷	2	$\frac{2cb-2cx}{x^2} = \frac{2cd-2cy}{y^2}$
2 reduc'd	3	$dx = by$. Q. E. I.

Here the Reader may obferve, that in the given Equation $\frac{\sqrt{x^2-c^2}}{\sqrt{x^2-c^2}}$, and $\frac{\sqrt{y^2-c^2}}{\sqrt{y^2-c^2}}$, the Numerator and Denominator are alike, and confequently deftroy each other, whence we have the fecond Step, and confequently free from Surds.

EXAMPLE.

Equation	1	$\frac{a}{\sqrt{1+a^2-\frac{a^2-x^2}{x^2}}} = b$
1 × x^2	2	$\frac{ax^2}{\sqrt{x^2+a^2x^2-a^2-x^4}} = bx^2$
2 ⊕ 2	3	$\frac{a^2x^4}{x^2+a^2x^2-a^2-x^4} = b^2x^4$
3 red. ±	4	$-b^2x^4 + \begin{matrix} a^2b^2 \\ +b^2 \\ -a^2 \end{matrix}\; x^2 = -a^2b^2$. Q. E. I.

EXAMPLE.

Suppofe $\frac{\sqrt[2]{yyy}-\sqrt[3]{yy}}{\sqrt[4]{y}} = \frac{\sqrt[3]{y}}{a}$, what's the Value of y? I fubftitute $x^{12}=y$, and then the Equation

becomes	1	$\frac{\sqrt[2]{x^{36}}-\sqrt[3]{x^{24}}}{\sqrt[4]{x^{12}}} = \frac{\sqrt[3]{x^{12}}}{a}$

 that

that is	2	$\frac{x^{18}-x^8}{x^3}=\frac{x^4}{a}$
Conseq.	3	$x^{11}-x=\frac{1}{a}$.

My Ingenious Friend and Mathematician, Mr. JOHN TURNER, in a Letter to me observes, "that Surds in "general, in the Solution of any Problem, the best Way "is to avoid them by substituting, so as to prevent them "coming into the Equation, as the above,

EXAMPLE.

Suppose $\frac{b^2x}{\overline{1-2xx}\sqrt{1-xx}}=\frac{b^2x}{\sqrt{1-xx}}+\frac{b^2s}{1-2xx}\sqrt{\frac{1+3xx}{1-xx}}$.

First, Make the Denominations all alike, by multiplying the Term of the Equation $\frac{b^2x}{\sqrt{1-xx}}$, *viz.* its Numerator by $\overline{1-2xx}$, and then $\frac{b^2x}{\overline{1-2xx}\sqrt{1-xx}}=\frac{b^2x-2b^2x^3}{\overline{1-2xx}\sqrt{1-xx}}+\frac{b^2s}{\overline{1-2xx}\sqrt{1-xx}}\sqrt{1+3xx}$. Now all the Denominators may be exterminated, being all the same, and the Equation becomes $b^2x=b^2x-2b^2x^3+b^2s\sqrt{1+3xx}$, or $2b^2x^3=b^2s\sqrt{1+3xx}$, involve this Equation, and it becomes $4b^4x^6=b^4s^2+3b^4s^2x^2$, or $x^6-\frac{3}{4}s^2x^2=\frac{1}{4}s^2$, and now put $z=x^2$, then $z^3-\frac{3}{4}s^2z=\frac{1}{4}s^2$. in its lowest Terms,

OPERATION.

OPERATION.

	1	$\frac{b^2x}{1-2xx\sqrt{1-xx}} = \frac{b^2x}{\sqrt{1-xx}} + \frac{b^2s}{1-2xx}\sqrt{\frac{1+3xx}{1-xx}}$
$1\times\overline{1-2xx}$	2	$\frac{b^2x}{1-2xx\sqrt{1-xx}} = \frac{b^2x-2b^2x^3}{1-2xx\sqrt{1-xx}} + \frac{b^2s}{1-2xx}\sqrt{\frac{1+3xx}{1-xx}}$
$2\div$	3	$b^2x=b^2x-2b^2x^3+b^2s\sqrt{1+3xx}$
$3\pm$	4	$2b^2x^3=b^2s\sqrt{1+3xx}$
4 ⊛ 2	5	$4b^4x^6=b^4s^2+3b^4s^2x^2$
$5\pm$	6	$4b^4x^6-3b^4s^2x^2=b^4s^2$
$6\div b^4$	7	$4x^6-3s^2x^2=s^2$
$7\div\overline{4}$	8	$x^6-\frac{3}{4}s^2x^2=\frac{1}{4}s^2$. Put $z=x^2$
Then	9	$z^3-\frac{3}{4}s^2z=\frac{1}{4}s^2$.

Q. E. I.

EXAMPLE.

Suppose $\frac{\overline{x+2b}^2}{x}\sqrt{b+x\times 4b} = b+\frac{\overline{x+2b}^2}{x}\sqrt{b+\frac{\overline{x+2b}^2}{x}\times 4b}\times\frac{2}{3}$

Let $n=\frac{2}{3}$, and the Equation order'd will be

$$\frac{x^2+4bx+4b^2}{x}\sqrt{4b^2+4bx} = \frac{x^2+5bx+4b^2}{x}\sqrt{\frac{4bx^2+20b^2x+16b^3}{x}}\times n.$$

Now x being expunged in the Denominator, and the Terms without the Radical Sign squared and multiplied by by the Terms under the Root, and the former Part of the Equation multiplied by x, and the latter by n^2, it will ſtand thus.

$$\left.\begin{matrix} x^4+8bx^3+24b^2x^2+ \\ 32b^3x+16b^4 \\ \text{Multiplied by } 4b^2x \\ +4bx^2 \end{matrix}\right\} = \left\{\begin{matrix} n^2x^4+10n^2bx^3+33n^2b^2x^2 \\ +40n^2b^3x+16n^2b^4 \\ \text{multiplied by } 4bx^2+ \\ 20b^2x+16b^3 \end{matrix}\right\} \text{ Now}$$

it is evident that it will produce an Equation of the 6th Power.

Example.

Suppoſe $\frac{b\sqrt{1-xx}}{1-2xx} = \frac{mc}{1-2xx} + \frac{mc}{2x\sqrt{1-xx}}$.

Or $\frac{b\sqrt{1-xx}-mc}{1-2xx} = \frac{mc}{2x\sqrt{1-xx}}$. Now multiply the firſt Part of the Equation by $2x\sqrt{1-xx}$, and the latter Part by $1-2xx$, and it will be

$$2bx-2bx^3-\overline{2mcx\sqrt{1-xx}} = mc-2mcx^2, \text{ or}$$

$$2mcx^2-2bx^3+2bx-mc = 2mcx\sqrt{1-xx}. \text{ By In-}$$

volution $4b^2x^6-8bmcx^5 \begin{matrix} +8m^2c^2 \\ -8b^2 \end{matrix} x^4+12bmcx^3 \begin{matrix} +4b^2 \\ -8m^2c^2 \end{matrix} x^2$

$-4mbcx+m^2c^2=0$.

Operation.

Equation	1	$\frac{b\sqrt{1-xx}}{1-2xx} = \frac{mc}{1-2xx} + \frac{mc}{2x\sqrt{1-xx}}$
$1-mc$	2	$\frac{b\sqrt{1-xx}-mc}{1-2xx} = \frac{mc}{2x\sqrt{1-xx}}$
$2\times 2x\sqrt{1-x^2}$	3	$\frac{2bx-2bx^3-\overline{2mcx\sqrt{1-xx}}}{1-2xx} = mc$
$3\times\overline{1-2x^2}$	4	$2bx-2bx^3-2mcx\sqrt{1-xx} = mc-2mcxx$
$4 \pm$	5	$2mcx^2-2bx^3+2bx-mc = 2mcx\sqrt{1-xx}$

5 ⊕ 2

5 ⊕ 2	6	$4b^2x^6 - 8bmcx^5 \begin{matrix} +8m^2c^2 \\ -8b^2x^4 \end{matrix} + 12bmcx^3 \begin{matrix} +4b^2 \\ -8m^2c^2 \end{matrix} x^2 - 4mbcx + m^2c^2 = 0.$

EXAMPLE.

Suppose $\frac{fa\sqrt{rr-xx}}{\sqrt{aa-2ax+rr}} = \frac{az\sqrt{rr-xx}}{\sqrt{zz+2zx+rr}}$, and you would find the Value of z, proceed thus.

Equation	1	$\frac{fa\sqrt{rr-xx}}{\sqrt{aa-2ax+rr}} = \frac{az\sqrt{rr-xx}}{\sqrt{zz+2zx+rr}}$
$1 \times \sqrt{a^2 - 2ax,}$ &c.	2	$fa\sqrt{rr-xx} = \frac{az\sqrt{rr-xx} \times \sqrt{aa-2ax+rr}}{\sqrt{zz+2zx+rr}}$
$2 \times \sqrt{z^2+2zx,}$ &c.	3	$fa\sqrt{rr-xx} \times \sqrt{zz+2zx+rr} = az\sqrt{rr-xx} \times \sqrt{aa-2ax+rr}$
$3 \div \sqrt{rr-xx}$	4	$fa\sqrt{zz+2zx+rr} = az\sqrt{aa-2ax+rr}$
$4 \div a$	5	$f\sqrt{zz+2zx+rr} = z\sqrt{aa-2ax+rr}$
5 ⊕ 2	6	$f^2 \times \overline{z^2+2zx+rr} = z^2 \times \overline{aa-2az+rr}.$

Q. E. I.

EXAMPLE.

Suppose	1	$c\sqrt{\frac{d^2a^2+d^2x^2}{mm} - x^2} = x\sqrt{b^2-c^2} + x\sqrt{\frac{d^2a^2+d^2x^2}{mm} - x^2}$
1 ±	2	$\overline{c-x}\sqrt{\frac{d^2a^2+d^2x^2}{mm} - x^2} = x\sqrt{bb-cc}$
2 ⊕ 2	3	$\overline{cc-2cx+xx} \times \frac{d^2a^2+d^2x^2}{mm} - x^2 = x^2 \times \overline{bb-cc}.$

3 ±

$3 \pm$	4	$x^4 - 2cx^3 = \dfrac{\begin{matrix} +ddcc \\ +d^2a^2x^2 - 2d^2a^2cx + d^2a^2c^2 \\ -m^2b^2 \end{matrix}}{dd+mm}$ $= 0.$

Hence by Form 14. Table of Theorems for Converging Series, the above Equation may easily be solv'd.

Q. E. I.

Sometimes Substitution renders the Work more easy, wherein an Equation is involved in Surds, as the following Examples will exhibit.

Example.

Equation	1	$\sqrt{b^2+3x^2} \times \sqrt{c^2+3x^2} \times a = 2x\sqrt{b^2-x^2}$ $+ 2x\sqrt{c^2-x^2}$
1 ⊚ 2	2	$\overline{b^2+3x^2} \times \overline{c^2+3x^2} \times a^2 = 4x^2 \times \overline{b^2-x^2}$ $+ 8x^2 \sqrt{c^2-x^2} \times \sqrt{b^2-x^2} + 4x^2$ $\times \overline{c^2-x^2}$
that is	3	$b^2c^2a^2 + 3c^2a^2x^2 + 3b^2a^2x^2 + 9a^2x^4 =$ $4b^2x^2 + 4c^2x^2 - 8x^4 + 8x^2\sqrt{c^2-x^2}$ $\times \sqrt{b^2-x^2}$
		Now for $b^2c^3a^2$, put m; for $3c^2a^2 + 3b^2a^2 - 4b^2 - 4c^2$, put $+n$; and for $9a^2+8$, put p
Then	4	$m+nx^2+px^4 = 8x^2\sqrt{c^2-x^2} \times \sqrt{b^2-x^2}$
4 ⊚ 2	5	$m^2 + 2mnx^2 + 2pmx^4 + n^2x^4 + 2pmx^6 +$ $p^2x^8 = 64x^4 \times \overline{c^2-x^2} \times \overline{b^2-x^2}.$

Hence by ordering the Terms you will have the Equation in the eighth Power.

EXAMPLE.

Suppose $\frac{\overline{uu}|^{\frac{1}{3}} + \overline{zzz}|^{\frac{1}{2}}}{\overline{z}|^{\frac{1}{4}}} = 3z + 3$, and $\frac{\overline{81z^3}|^{\frac{1}{2}} - 8u^2}{3u}$ $= \sqrt{z}$, or $\frac{\sqrt[3]{uu} + \sqrt{zzz}}{\sqrt[4]{z}} = 3z + 3$, and $\frac{\sqrt{81z^3} - 8u^2}{3u} = \sqrt{z}$. Which are the same Equations, but differently set down, this being the old, the other the new Way of Notation, to find the Values of u and z? Subst. y^3, for u and x^4 for z.

Then	1	$\frac{\sqrt[3]{y^6} + \sqrt{x^{12}}}{\sqrt[4]{x^4}} = 3x^4 + 3$
And	2	$\frac{\sqrt{81x^{12}} - 8y^6}{3y^3} = \sqrt{x^4}$
1 ພ	3	$\frac{y^2 + x^6}{x} = 3x^4 + 3$
$3 \times x$	4	$y^2 + x^6 = 3x^5 + 3x$
2 ພ	5	$\frac{9x^6 - 8y^6}{3y^3} = x^2$
$5 \times 3y^2$	6	$9x^6 - 8y^6 = 3y^3x^2$.

Here are two unknown Quantities, and two Equations, by which it will be easy to find the Value of each by the Rules already laid down, *viz.* $y = x = 3$.

There are a great many Cases besides, which may by the Judgment of the Algebraist, from what I have laid down, be contracted, or reduced lower, or exterminated by Substitution, which cannot be brought under any Rule, and can only come by frequent Practice.

I think I have ſaid what is neceſſary to enable the Reader, with little Practice, to ſolve any Equation analytically in the moſt conciſe and elegant Manner. I ſhall deſiſt giving any more Examples, and make a Tranſition to the other Part, how to ſolve any adfected Equation into Numbers (after they have been order'd according to our Method aforeſaid) by an univerſal Method of *Converging Series.*

AN UNIVERSAL METHOD OF *Converging Series.*

Handled in a very easy, plain and expeditious Method.

DEFINITION.

A Series which approaches continually to the Truth, is said to converge, and which continually goes from it is said to diverge.

COROLLARY.

Therefore a Series of Fractions continually decreasing are converging, but others whose Terms continually increase are diverging.

Now in all Equations higher than a *Quadratic* (if affected) the best Way is to solve the same by a Recourse had to *Infinite* or *Converging Series*, and the common Method, that which I call the most easy, assume $m+n$ for the Value of your unknown Quantity, that is assume $m =$ Root of your Equation, as near as you can (tho' if you assume never so far from the true Root, yet it will by renewing the Operation converge to it) and assuming $+$ or $-n$ for the Deficiency, then it will be $m+n$, or $m-n =$ Root. Therefore for the Usefulness of Dispatch I have raised the following Table to the 8th Power upon the above Assumption,

 that

that the Learner may at first View substitute his Equation aright.

N. B. It is here to be observed, that it is needless to have n in your Equation above the Square, which the following Table exhibits.

The TABLE *of* POWERS.

		The	
$m+n$	- - - - - - -	Simple Power	x
$m^2+2mn+n^2$	- - -	The Square	$=x^2$
$m^3+3m^2n+3mn^2$	- -	Cube	$=x^3$
$m^4+4m^3n+6m^2n^2$	- -	4th Power	$=x^4$
$m^5+5m^4n+10m^3n^2$	-	5th Power	$=x^5$
$m^6+6m^5n+15m^4n^2$	-	6th Power	$=x^6$
$m^7+7m^6n+21m^5n^2$	-	7th Power	$=x^7$
$m^8+8m^7n+28m^6n^2$	-	8th Power	$=x^8$

N. B. The Table of Powers above are fitted for Operation, if your Equation be Affirmative, but if Negative, change the Signs, as the following Examples will shew.

Given this Equation to find the Value of y.

EXAMPLE I.

$$y^3-360y^2+43200y-1600000=0.$$

First, Suppose $m=$ Root of y nearly, and let n be the Defect, that is,

Let $m+n=y$ }
Then $y=m+n$ } Which Values substituted in the given Equation, rejecting all the
$y^2=m^2+2mn+n^2$ }
$y^3=m^3+3m^2n+3mn^2$ }

Terms wherein the Dimensions of n are above the Square, we have

$m^3+3m^2n+3mn^2-360m^2-720mn-260n^2+43200m+43200n-1600000=0.$

Transpose all the m's on one Side the Equation, $\therefore$ $3m^2n+3mn^2-720mn-360n^2+43200n=-m^3+360m^2-43200m+1600000.$

Now

Now take all the Terms wherein the Power of n is singly concerned. Then arises this general

THEOREM.

$$n = \frac{-m^3 + 360m^2 - 43200m + 1600000}{3m^2 + 3mn - 720m - 360n + 43200}.$$

Now suppose $m =$ to some Number, which let it be as near the Root as possibly can, which here I suppose $= 60$.

Collect the Terms + and −.

$$\text{Then } \left.\begin{array}{r} -m^3 = -216000 \\ -43200m = -2592000 \end{array}\right\} \quad \begin{array}{r} +360m^2 = +1296000 \\ + \qquad +1600000 \end{array}$$

$$-2808000 \qquad\qquad +2896000$$

Now $2896000 - 2808000 = 88000$ for a Dividend. Then take all the Terms wherein n is not concern'd, and put into Numbers in the Divisor. Thus,

$$\left.\begin{array}{r} 3m^2 = 10800 \\ 43200 \end{array}\right\} \qquad -720m = -43200$$

Now $54000 - 43200 = 10800$ Divisor.

Therefore 10800) 88000 ($8 = n$ nearly.

Now take those Powers where n is concerned,

$$\text{As } \left.\begin{array}{r} 3mn = 1440 \\ -360n = -2880 \end{array}\right\} 1440 - 2880 = -1440, \text{ which}$$

must be taken from the last Divisor (by Reason of unlike Signs)

$$\begin{array}{r} 10800 \\ -1440 \\ \hline 9360 \end{array}$$

9360) 88000 ($9 = n$ more nearly.

Consequently $y = m + n = 69.401$.

Here the Learner may see,

That after I had noted down my Equation for *Converging Series*, first I consider well my Equation, the second Power proving a Negative, I must make all the Squares Negative from my Table, because my Equation is Negative.

Then

Then I assume $m+n=$ the Root: I raise it to the Power of the Equation, which here is a Cubic One, and after it is rightly substituted, the next I transpose all the m's on one Side the Equation, now seeing I have got all my m's on one Side the Equation, I make that Side a Dividend, and then take n from all those Quantities where it is simply concerned, and put $n=$ Dividend of the m's, and the Divisor will be freed from n, where it was simply concern'd, and those Quantities, where it was concern'd in the Square is brought down to a single n,

As $3m^2n+3mn^2-720mn-360n^2+43200n=-m^3+360m^2-43200m+1600000$.

Here you see I have got all the m's on one Side the Equation, which I make a Dividend, then I take n from all those Quantities where it is simply concern'd, as $3m^2n$, $-720mn$, $+43200n$, and it makes $3m^2$, $-720m$, $+43200$, which n I place thus,

$n=\frac{-m^3+360m^2-43200m+1600000}{3m^2+3mn-720m-360n+43200}$, which is call'd a Theorem.

Here the Learner may see how the n's vanish'd out of the Divisor, where it was simply concern'd, and where the Square was multiplied in the Quantities, is reduced to the simple Power of n, as $3mn$, $-360n$.

And so of any other.

There be several Things to be observed in this Method of *Converging Series*, *viz.* That at each Operation, the Converging Number n will double the last preceding m (or Numbers of Figures in the last Root, *especially after the second Operation*,) the Imperfection being only in the last Figure of the Root, so increased, which often proves too large, and therefore consequently the next converging Number n will have the Negative Sign —.

Also if there happen to be a Mistake committed in any Operation, such Mistake doth not destroy the preceeding Work, for the same will be rectified (tho' it be not discovered) in the next succeeding Operation, unless it be very gross.

Again it produceth the Roots of all Powers, be they never so high, and in the same Manner, and with the

same

ſame Exactneſs, as it doth thoſe of a lower Rank, the reſpective Involutions being conſider'd, which require to be always of the ſame Height with the given Powers, and the Diviſor of the next inferior or lower Powers.

Suppoſe I had made $m = 70$ (more than its real Value) then the above Theorem in Numbers would ſtand thus,

$$\left.\begin{array}{rl} -m^3 = & -343000 \\ +360m^2 = & +1764000 \\ -43200m = & -3024000 \\ & +1600000 \end{array}\right\}$$

Signs collected.

$$\begin{array}{r} +1764000 \\ +1600000 \\ \hline +3364000 \end{array} \qquad \begin{array}{r} -3024000 \\ -343000 \\ \hline -3367000 \end{array}$$

Then $+3367000-3364000=-3000=$ Dividend.

$$\left.\begin{array}{rl} \text{And}\quad 3m^2 = & +14700 \\ -720m = & -50400 \\ & +43200 \end{array}\right\}$$

That is the Signs collected.

$$\begin{array}{r} +14700 \\ +43200 \\ \hline 57900 \end{array}$$ And $57900-50400 = 7500$ Diviſor.

Then $7500)\underline{-3000.0}\,(-.4=n$ nearly;

Then $3mn = -84$,
And $-360n = 144$. Therefore $144 - 84 = 60$ to be added;

Conſequently $\begin{array}{r} 7500 \\ +\underline{\quad 60} \end{array}$

$7560)-3000.0000\,(-.3968 = n$ more nearly.

Thence

Thence it follows that $m = 70$.

And $-n = -.3968$

And therefore $m - n = y = 69.6032$

Q. E. I.

See Form 2d. in the Table of Theorems.

EXAMPLE 2.

Suppose $8x^3 - 1440x^2 + 86400x - 1600000 = 0$, Quere the Value of x?

Put $m + n = x$,

Then by the Table $8x^3 = 8m^3 + 24m^2n + 24mn^2$; and the Equation after you have substituted right becomes,

$8m^3 + 24m^2n + 24mn^2 - 1440m^2 - 2880mn - 1440n^2 + 86400m + 86400n - 1600000 = 0$.

Transpose all the m's, and it is

$24m^2n + 24mn^2 - 2880mn - 1440n^2 + 86400n = -8m^3 + 1440m^2 - 86400m + 1600000$.

THEOREM.

$$n = \frac{-8m^3 + 1440m^2 - 86400m + 1600000}{24m^2 + 24mn - 2880m - 1440n + 86400}$$

Now having got the above Theorem by Transposition, and taking all the Terms where the single Power of n is concern'd. My next Work is to assume $m =$ to a Number, as near the Root as I can, which, considering the Equation, I at a venture assume $m = 30$; then the above Theorem in Numbers will stand thus,

$$n = \frac{-216000 + 1296000 - 2592000 + 1600000}{21600 - 86400 + 86400}$$

That is,

$$n = \frac{88000}{21600} = 4 = n \text{ nearly};$$

Now

Now multiply thofe Numbers where n is concern'd by its Value; as $24mn = 2880$

$-1440n = -5760$

and $-5760 + 2880 = -2880$, which muft be taken from the Divifor, by Reafon of unlike Signs.

As 21600
—2880
18720) 88000.00 (4. 7, more nearly,

Confequently $m + n = x = 34.7$.

(*See Form 2d in the Table.*) Q. E. I.

EXAMPLE 3.

Suppofe $900000x - x^3 = 243000000$

Affume $m + n = x$,

Then according to the Table of Powers, the Equation is

$$900000m + 900000n - m^3 - 3m^2n - 3mn^2 = 243000000.$$

Tranfpofed,

Is $900000n - 3m^2n - 3mn^2 = 243000000 + m^3 - 900000m$. Now taking all the Powers where n is fimply concerned, we get this

THEOREM.

$$n = \frac{243000000 + m^3 - 900000m}{900000 - 3m^2 - 3mn}.$$

Now I affume m as near the Root as I can, which I guefs $= 250$ at a venture.

Then $m^3 = 15625000$
$+ \quad 243000000$
258625000
And $-900000m = -225000000$
$33625000 =$ Dividend.

Then $-3m^2 = -187500$, which taken from 900000, Leaves 712500 for the Divifor.

OPERATION.

712500) 33625000 (47 $= n$ nearly.

Then $-3mn = -35250$ to be taken from the Divisor, by Reason of unlike Signs. Thus,

$$\begin{array}{r} 712500 \\ -\ 35250 \\ \hline 677250 \end{array}$$

677250) 33625000 (49.64 $= n$ more nearly;
Consequently $m + n = x = 299.64 = 300$ *ferè*.

(See Form 11th.)

EXAMPLE 4.

Suppose $x^3 - 50x = 120$.

By assuming $m + n = x$, then, according to the Table, will be $m^3 + 3m^2n + 3mn^2 - 50m - 50n = 120$.

Transposed, is

$3m^2n + 3mn^2 - 50n = -m^3 + 50m + 120$.

Then by taking the single Power of n, we get this

THEOREM.

$$n = \frac{120 + 50m - m^3}{3m^2 + 3mn - 50}$$

Now let us make $m = 10$, then $-m^3 = -1000$, and $+ 50m = 500$
$+ 120 = 120$
620 Then 620 taken from -1000, leaves -380 for the Dividend.

Then $3m^2 = 300$
and $- 50$
leaves 250 for a Divisor.

Consequently,

250) -380 ($-1.5 = n$ nearly.

Then $3mn = -45$ to be subtracted from the Divisor by Reason of unlike Signs.

250

$$\begin{array}{r} 250 \\ -45 \\ \hline 305) -380 \end{array} (-1.85 = n \text{ more nearly.}$$

Consequently $m - n = x = 8.15$.

Q. E. I.

(See Form 10th.)

EXAMPLE 5.

Suppose $x^3 + x^2 + 43x = 1197$.

Assume $m + n = x$. Then raising $m + n$ to the Power of the Equation, it will stand thus,

$$m^3 + 3m^2n + 3mn^2 + m^2 + 2mn + n^2 + 43m + 43n = 1197.$$

Transpos'd,

$$3m^2n + 3mn^2 + 2mn + n^2 + 43n = 1197 - m^3 - m^2 - 43m.$$

Hence arises this

THEOREM.

$$n = \frac{1197 - m^3 - m^2 - 43m}{3m^2 + 3mn + 2m + n + 43}.$$

Now suppose $m = 10$. Then

$$\begin{array}{rcr} -m^3 & = & -1000 \\ -m^2 & = & -100 \\ -43m & = & -430 \\ \hline & & -1530 \end{array}$$

And $-1530 + 1197 = -333$ Dividend.

$$\begin{array}{rcr} \text{And } 3m^2 & = & 300 \\ 2m & = & 20 \\ + & & 43 \\ \hline & & 363 \end{array}$$

Hence $363) -333.0 (-.9 = n$ nearly.

$$\begin{array}{rcl} \text{Then } -3mn & = & -27 \\ -n & = & -\ \ .9 \\ \hline & & -27.9, \end{array}$$ to be taken from the Divisor.

 Thus

Thus 363
-27.9

$335.1) -333.0000 (-.993 = n$ more nearly.

Consequently I took m too much, and therefore I must deduct n from it; and therefore $10 - .993$ is $= m - n = x = 9.007$. Q. E. I.

(See Form 1st.)

EXAMPLE 6.

Given $y^3 - 21197y = -398439$.

Assume $m + n = y$, then the Equation being raised, or taken from the Table of Powers, will stand thus,

$$m^3 + 3m^2n + 3mn^2 - 21197m - 21197n = -398439.$$

Transpose all the m's, and it is

$$3m^2n + 3mn^2 - 21197n = -398439 - m^3 + 21197m.$$

Consequently arises this

THEOREM.

$$n = \frac{-398439 - m^3 + 21197m}{3m^2 + 3mn - 21197}$$

Assume $m = 130$ at a venture, then $-m^3 = -2197000$
and $-\ \ 398439$
-2595439

And $21197m = 2755610$.

Consequently $2755610 - 2595439 = 160171$, a Dividend

Then $3m^2 = 50700$, and $50700 - 21197 = 29503$.

Therefore $29503)\ 160171\ (5 = n$ nearly.

Then $+3mn = 1950$ to be added to the Divisor.

Thus 29503
1950

$31453)\ 160171\ (5.09 = n$ more nearly.

Consequently $m + n = y = 135.09$.

Q. E. I.

(See Form 10th.)

EXAMPLE 7.

Given $x^3-240x^2-241x=-14214$. Quere x?

Put $m+n=x$, then according to my Table of Powers, the Equation is

$m^3+3m^2n+3mn^2-240m^2-480mn-240n^2-241m-241n=-14214$.

Transpos'd,

$3m^2n+3mn^2-480mn-240n^2-241n=-m^3+240m^2+241m-14214$.

Then taking the Simple Power of n, we have this

THEOREM.

$$n=\frac{-m^3+240m^2+241m-14214}{3m^2+3mn-480m-240n-241}$$

Now suppoſing $m=10$, the above Theorem in Numbers will ſtand thus,

$$n=\frac{-1000+24000+2410-14214}{300-4800-241}=\frac{11196}{-4741}=-2$$

$=n$ nearly.

Then $+3mn=-60$
$-240n=480$. Conſequently $480-60=420$.

And -4741
$+420$
$-4321)\ 11196\ (-2.59=n$ more nearly.

Therefore $m-n=x=7.41$. Q. E. I.

(See Form 3d.)

EXAMPLE 8.

Suppoſe $x^3+81128x=421824$.

Aſſume $m+n=x$, then raiſing the Equation to the third Power, or taking the third Power out of the Table,

Table, and ſubſtituting the Equation it will become $m^3+3m^2n+3mn^2+8112m+8112n=421824$.

By tranſpoſing all the m's, it will be

$3m^2n+3mn^2+8112n=421824-m^3-8112m.$

Hence ariſes this

THEOREM.

$$n=\frac{421824-m^3-8112m}{3m^2+3mn+8112}.$$

Here let us ſuppoſe $m=40$, then the above Theorem in Numbers will ſtand thus,

$n=\frac{33344}{12912+120n}$. That is

$12912)\ 33344\ (2=n$ nearly.

Then $120n=240$ to be added to the Diviſor, by Reaſon of like Signs.

As 12912
  240
$13152)\ 33344\ (2.535=n$ more nearly.

Conſequently $m+n=x=42.535$. Q. E. D.

(See Form 9th.)

EXAMPLE 9.

$x^3+x=1$. Quere x?

Now by aſſuming $m+n=x$, and raiſing the ſaid $m+n$ to the Equation, it will ſtand thus,

$m^3+3m^2n+3mn^2+m+n=1.$

Tranſpos'd.

$3m^2n+3mn^2+n=1-m^3-m.$

Then according to the Equation there ariſes this

THEOREM.

THEOREM.

$$n = \frac{1 - m^3 - m}{3m^2 + 3mn + 1},$$

Here considering our Equation as Unity (every one that has the least Idea of Fractions knows, that a Fraction multiply'd by a Fraction decreases the Value) Therefore I assume $m = .9$; then

$$\begin{array}{rl} -\quad m^3 = & -.729 \\ m = & -\quad .9 \\ \hline & .728, \end{array}$$

and $1 - .738 = .262$, a Dividend, and $3m^2 = 2.43 + 1 = 3.43$, a Divisor.

Therefore it will be

$3.43)\ .262\ (.076 = n$ nearly.

And now $3mn = .2052$. Consequently must be added to the Divisor by Reason of like Signs, it is

$$\begin{array}{l} 3.43 \\ \ .2052 \\ \hline 3.6352)\ .262\ (.07207 = n \text{ more nearly.} \end{array}$$

Consequently therefore $m + n = x = .97207 = 1.$ nearly.

(*See Form 9th.*)

EXAMPLE 10.

Given $y^3 + 6272y = 288512$.

By assuming $m + n = y$, and raising it to the Equation, it is $= m^3 + 3m^2n + 3mn^2 + 6272m + 6272n = 288512$.

By Transposition it becomes

$$3m^2n + 3mn^2 + 6272n = 288512 - m^3 - 6272m.$$

From which arises this

THEOREM.

$$n = \frac{288512 - m^3 - 6272m}{3m^2 + 3mn + 6272}.$$

Now

Now by assuming $m = 30$, the above Theorem in Numbers $n = \frac{73352}{8972 + 90n} = 8 = n$ nearly.

Then taking the Value of n in $90n$, and it is $= 720$, which must be added to the Divisor, and it will be

$8972 + 720 = 9692$, and

$9692)\ 73352\ (7.567 = n$ more nearly.

Consequently $m + n = y = 37.567$.

Q. E. I.

(See Form 9th.)

EXAMPLE II.

Given $x^3 - 171.91x^2 + 7905.6x = 71460$.

By assuming $m + n = x$, and raising it to the Power of the Equation, it will be

$m^3 + 3m^2n + 3mn^2 - 172m^2 - 344mn - 172n^2 + 7905m + 7905n = 71460$.

By Transposition

Is $3m^2n + 3mn^2 - 172n^2 - 344mn + 7905n = 71460 - m^3 + 172m^2 - 7905m$.

THEOREM.

$n = \frac{71460 - m^3 + 172m^2 - 7905m}{3m^2 - 3mn - 344m - 172n + 7905}$. Now by making $m = 10$, it will be in Numbers,

$n = \frac{71460 - 1000 + 17200 - 79050}{300 + 30n - 3440 - 172n + 7905} = \frac{8610}{4765} = 1.$ $= n$ nearly; then according to the second Operation, we get $n = 1.862$.

Consequently $m + n = x = 11.862$.

Q. E. I.

(See Form 2d.)

EXAMPLE.

EXAMPLE 12.

Given $x^3 - 6\frac{109}{144}x = 2\frac{31}{144}$. Quere x.

Reduced to a common Denominator

is $\frac{144x^3 - 973x}{144} = \frac{319}{144}$;

and by multiplying each Part by 144, it will become

$144x^3 - 973x = 319$ Equation

Assume $m + n = x$; then according to the Nature of the Equation it is

$144m^3 + 432m^2n + 432mn^2 - 973m - 973n = 319.$

Transposed,

Is $432m^2n + 432mn^2 - 973n = 319 - 144m^3 + 973m.$

Hence arises this

THEOREM.

$$n = \frac{319 - 144m^3 + 973m}{432m^2 + 432mn - 973}.$$

Suppose $m = 2$, then by putting the above Theorem in Numbers and Division, we get $1 = n$ for the first Operation, and the second Operation we get $n = .6873$.

Consequently $m + n = x = 2.6873.$

Q. E. I.

(See Form 10th.)

EXAMPLE 13.

$x^3 - 96x^2 = 6600\ 458.\ 365090.$

Assume $m + n = x$.

Then according to my Table of Powers it will be, *viz.*

$m^3 + 3m^2n + 3mn^2 - 96m^2 - 192mn - 96n^2 = 6600458.365.$

By Transposition.

$3m^2n + 3mn^2 - 192mn - 96n^2 = 6600458.365090 - m^3 + 96m^2$.

Hence arises this

THEOREM.

$$n = \frac{6600458.365090 - m^3 + 96m^2}{3m^2 + 3mn - 192m - 96n}.$$

Hence by assuming $m = 200$, we get $n = 25.6404$.

Consequently $m + n = x = 225.6404$.

(See Form 7th.)

EXAMPLE 14.

Given $x^3 - 44100x + 176400 = 0$.

Assume $m + n = x$, then according to the Nature of the Equation, it will be

$$m^3 + 3m^2n + 3mn^2 - 44100m - 44100n + 176400 = 0.$$

Then by Transposition.

$$3m^2n + 3mn^2 - 44100 = -m^3 + 44100m - 176400.$$

From which arises this

THEOREM.

$$n = \frac{-m^3 + 44100m - 176400}{3m^2 + 3mn - 44100}.$$

Here let us take $m = 3$, then the above Theorem in Numbers is $n = \dfrac{-44127}{-44073 + 9n} = 1 = n$ nearly.

Then $9n = 9$. to be taken from the Divisor, *viz.* $-44073 + 9 = -44064$.

And $-44064) -44127$ ($1.0001429 = n$ more nearly.

Consequently $m + n = x = 4.0001429$.

Q. E. I.

(See Form 10th.)

EXAMPLE,

EXAMPLE 15.

Given $x^3 + 438x^2 - 7825x - 98508430 = 0.$

By assuming $m + n = x$, the Equation becomes

$m^3 + 3m^2n + 3mn^2 + 438m^2 + 876mn + 438n^2 - 7825m - 7825n - 98508430 = 0.$

By Transposition.

$3m^2n + 3mn^2 + 876mn + 438n^2 - 7825n = -m^3 - 438m^2 + 7825m + 9850843.$

Hence arises this

THEOREM.

$$n = \frac{-m^3 - 438m^2 + 7825m + 98508430}{3m^2 + 3mn + 876m + 438n - 7825}.$$

Here let us assume $m = 300$.

Then $-m^3 = -27000000$ $\quad$ $+7825m = 2347500$

$-438m^2 = -39420000$ $\quad$ and $+98508430$

-66420000 $\quad$ 100855930

Now $100855930 - 66420000 = 34435930$ = Dividend.

And $+3m^2 = 270000$

$+876m = 262800$

532800. And $532800 - 7825 = 524975$ = the Divisor.

That is $524975)\ 532800\ (65 = n$ nearly.

Then $3mn = 58500$

$438n = 28470$

86970, and $86970 - 7825 = 79145$ to be added to the Divisor by Reason of like Signs. As

524975

79145

$604120)\ 34435930\ (57.0018 = n$ more nearly.

Consequently $m + n = x = 357.0018$, Q. E. I.

(See Form 3d.)

 EXAMPLE

EXAMPLE 16.

Given $x + xx + xxx = 29791000$.
By assuming $m + n = x$, the Equation becomes

$$m + n + m^2 + 2mn + n^2 + m^3 + 3m^2n + 3mn^2 = 29791000.$$

By Transposition it will be

$$n + 2mn + n^2 + 3m^2n + 3mn^2 = 29791000 - m - m^2 - m^3.$$

Hence arises this

THEOREM.

$$n = \frac{29791000 - m - m^2 - m^3}{3m^2 + 3mn + 2m + 1 + n}, \text{ or}$$

$$n = \frac{29791000 - m - m^2 - m^3}{n + 1 + 2m + 3mn + 3m^2}, \text{ which is all the same:}$$

It is no Matter how the Terms stand, so they be duly collected. And then by assuming $m = 300$, the above Theorem put into Numbers, we shall find the Value of $n = 10$. Consequently $m + n = x = 310$.

Which was to be found.

(See Form 1.*)*

Biquadratic

Biquadratic Æquations.

EXAMPLE 17.

GIVEN $x^4+x^3+x^2+x = 20736$,
or $x+x^2+x^3+x^4 = 20736$,
or $x^4+x^3+x^2+x - 20736 = 0$, which are all the ſame, required the Value of x, here according to my former Examples. I aſſume $m+n=x$. Then according to my Table of Powers, the Equation becomes

$$m^4+4m^3n+6m^2n^2+m^3+3m^2n+3mn^2+m^2+2mn+n^2+m+n = 20736.$$

Now tranſpoſe all the m's on one Side the Equation, and it will be

$$4m^3n+6m^2n^2+3m^2n+3mn^2+2mn+n^2+n = -m^4-m^3-m^2-m+20736.$$

Hence by taking all the Terms where the ſimple Power of n is concern'd (as you did in the Cubics) you'll get this

THEOREM.

$$n = \frac{-m^4-m^3-m^2-m+20736}{4m^3+6m^2n+3m^2+3mn+2m+1+n}.$$

Now let us take $m=10$.

Then $-m^4 = -10000$
$-m^3 = -1000$
$-m^2 = -100$
$-m = -10$
-11110. And $20736 - 11110 = 9626$ divided.

And

And $4m^3 = 4000$
$3m^2 = 300$
$2m = 20$
$+ 1$

4321 Divisor.

4321) 9626 (2 $= n$ nearly.

Then $6m^2n = 1200$
$3mn = 60$
$n = 2$

1262 to be added to the Divisor.

As 4321
1262

5583) 9626 (1.724 $= n$ more nearly.

Consequently $m+n = 11.724 = x$. 12 *ferè.*

(See Form 12th.)

EXAMPLE 18.

Given $x^4+40x^3+751x^2-9000x = 9000$.

Assume $m+n = x$.

Then by the Table of Powers, the Equation will be

$m^4+4m^3n+6m^2n^2+40m^3+120m^2n+120mn^2+751m^2+1502mn+751n^2-9000m-9000n = 9000$.

Transpos'd.

$4m^3n+6m^2n^2+120m^2n+120mn^2+1502mn+751n^2-9000n = -m^4-40m^3-751m^2+9000m+9000$.

And by collecting the single Power of n, we get this

THEOREM.

$$n = \frac{9000+9000m-751m^2-40m^3-m^4}{4m^3+6m^2n+120m^2+120mn+1502m+751n-9000}$$

Now let us suppose $m = 10$, then the above Theorem will in Numbers stand thus, *viz.*

$n =$

$$n = \frac{54900}{22020 + 600n + 1200n + 751n} = 2 = n \text{ nearly.}$$

And then taking the Value of n, and working for a second Operation, we get $m+n=x=12.00570103$.

Q. E. I.

(*See Form 14th.*)

EXAMPLE 19.

Given $-x^4+845.77x^3-220744.848x^2+36854112.9x=6192379528.849$.

By assuming $m+n=x$, we shall have from my Table of Powers this following one, *viz.*

$-m^4-4m^3n-6m^2n^2+845.77m^3+2537.31m^2n+2537.31mn^2-220744.848m^2-441489.696mn-220744.848n^2+36854112.9m+36854112.9n=6192379528.849$.

Transpos'd.

$-4m^3n-6m^2n+2537.31m^2n+2537.31mn-441489.696mn-220744.848n+36854112.9n=m^4-845.77m^3+220744,848m^2-36854112.9m+6192379528.849$.

Hence arises from what has been done this

THEOREM.

$$n = \frac{m^4-845.77m^3+220744.848m^2-36854112.9m+6192379528.849}{-4m^3-6m^2n+2537.31m^2+2537.31mn-441489.696m-220744.848n+36854112.9}.$$

Now seeing there are Decimals in the above Theorem, which causeth a great deal of Trouble in the Operation; therefore to contract the Work there is no need of such a Nicety, it may be expressed thus, as a neater

THEOREM.

THEOREM.

$$n = \frac{m^4 - 845m^3 + 220740m^2 - 36854000m + 6192379528}{-4m^3 - 6m^2n + 2537m^2 + 253700mn - 441500m - 220700n + 36854113}.$$

By aſſuming $m = 300$, the Value will very eaſily be found from what has been delivered above.

(*See Form* 15*th*.)

EXAMPLE 20.

Given $x^4 - 3x^2 + 75x = 10000$.

Aſſume $m + n = x$, then by the Table of Powers the above Equation will become

$m^4 + 4m^3n + 6m^2n^2 - 3m^2 - 6mn - 3n^2 + 75m + 75n = 10000$

Tranſpos'd is.

$4m^3n + 6m^2n^2 - 6mn - 3n^2 + 75n = 10000 - m^4 + 3m^2 - 75m$.

Hence ariſes this

THEOREM.

$$n = \frac{10000 - m^4 + 3m^2 - 75m}{4m^3 + 6m^2n - 6m - 3n + 75}.$$

Now by aſſuming $m = 10$, then the above Theorem in Numbers, *viz.*

$$n = \frac{-450}{4015 + 600n - 3n} = .11 = n \text{ nearly.}$$

Then taking the Value of n, and we ſhall have -66.33 to be taken from the Diviſor; as

4015
—66.33
3948.67) —450 (—.11396 = n more nearly.

Whence

Whence it follows that m was taken too great; and therefore $m-n=x=9.88604$. the true Root.

Q. E. I.

EXAMPLE 21.

$y^4-4y^3=13824$.

By assuming $m+n=y$, and taking the Power of the Equation from the Table of Powers, it will be

$$m^4+4m^3n+6m^2n^2-4m^3-12m^2n-12mn^2=13824.$$

By Transposition we have

$$4m^3n+6m^2n^2-12m^2n-12mn^2=13824-m^4+4m^3.$$

Thence arises this

THEOREM.

$$n=\frac{13824-m^4+4m^3}{4m^3+6m^2n-12m^2-12mn}.$$

By assuming $m=10$, the above Theorem in Numbers is

$$n=\frac{13824-1000+4000}{4000+600n-1200-120n}=\frac{7824}{2800}=2=n$$

nearly.

Then taking those Powers where n is concern'd, and multiply'd by 2 (the Value of n nearly just now found) the Products is 960 to be added to the Divisor, *viz.*

$$\begin{array}{l} 2800 \\ \underline{960} \\ 3760)\ 7824\ (2.08085=n \text{ more nearly.} \end{array}$$

Consequently $m+n=y=12.08085$.

Q. E. I.

EXAMPLE 22.

$x^4-8x^3+20x^2-15x+.5=0$.

Q By

By assuming $m+n=x$; and substituting the Equation from the Table of Powers, we shall have this Equation, viz.

$$m^4+4m^3n+6m^2n^2-8m^3-24m^2n-24mn^2+20m^2+40mn+20n^2-15m-15n+.5=0.$$

By Transposition we have

$$4m^3n+6m^2n^2-24m^2n-24mn^2+40mn-15n=-m^4+8m^3-20m^2+15m-.5.$$

From whence arises this

THEOREM.

$$n=\frac{-m^4+8m^3-20m^2+15m-.5}{4m^3+6m^2-24m^2-24mn+40m-15}.$$

And now seeing the Equation is a Fraction, suppose $m=1$. Then it is evident that the Theorem in Numbers is

$$n=\frac{1.5}{5.6+6n-24n}=.26=n \text{ nearly}.$$

And then taking the Quantities where n is, and it is -4.68, which must be taken from the Divisor by Reason of unlike Signs; as

$$\begin{array}{r} 5.6 \\ -4.68 \\ \hline .92)\ 1.5\ (1.630434=n \text{ more nearly}. \end{array}$$

Consequently $m+n=1.630434=x$.

Q. E. I.

(*See Form* 15*th.*)

EXAMPLE 23.

$$x^4+2x^3-288x^2-506x=1513.$$

By taking $m+n=x$, and their Power substitute the given Equation from the Table of Powers,

$m^4+4m^3n+6m^2n^2+2m^3+6m^2n+6mn^2-288m^2-576mn-288n^2-506m-506n=1513.$

By Transposition it is

$4m^3n+6m^2n^2+6m^2n+6mn^2-576mn-288n^2-506n=-m^4-2m^3+288m^2+506m+1513.$

From which arises this

THEOREM.

$$n=\frac{1513-m^4-2m^3+288m^2+506m}{4m^3+6m^2n+6m^2+6mn-576m-288n-506}.$$

And let us make $m=20$, then the abovesaid Theorem in Numbers, and divided, we shall find that we have taken m too great, therefore we shall find n a Negative Quantity, which must be subtracted from m, and we shall find the Value of $m-n=x=17$.

Q. E. I.

(*See Form 15th.*)

EXAMPLE 24.

$x^4+x=126\frac{64}{81}.$

Reduc'd.

$81x^4+81x=10270.$

Assume $m+n=x$, then by the Table of Powers, the Equation is

$81m^4+324m^3n+486m^2n^2+81m+81n=10270.$

Transposed is

$324m^3n+486m^2n^2+81n=10270-81m^4-81m.$

From which arises this

THEOREM.

$$n=\frac{10270-81m^4-81m}{324m^3+486m^2n+81}.$$

 By

By assuming $m = 3$, the Theorem in Numbers for the first Operation, *viz.*

8829) 3466 (.3 $= n$ nearly.

Then $486m^2n = 1312.2$ to be added to the Divisor, *viz.*

$$\begin{array}{l} 8829 \\ \underline{1312.2} \\ 10141.2) \; 3466 \; (.34177 = n \text{ more nearly.} \end{array}$$

Consequently $m + n = x = 3.34177$.

Q. E. I.

EXAMPLE 25.

Given $x^4 + 42.x^3 - 420x^2 - 1822x - 1799.725 = 0$.

Assume $m + n = x$, and then by my Table of Powers it will be

$m^4 + 4m^3n + 6m^2n^2 + 42m^3 + 126m^2n + 126mn^2 - 420m^2 - 840mn - 420n^2 - 1822m - 1822n - 1799.725 = 0$.

By Transposition.

$4m^3n + 6m^2n^2 + 126m^2n + 126mn^2 - 840mn - 420n^2 - 1822n = 1799.725 - m^4 + 420m^2 + 1822m$.

From which arises this

THEOREM.

$$n = \frac{1799.725 - m^4 + 420m^2 + 1822m}{4m^3 + 6m^2n + 126m^2 + 126mn - 840m - 420n - 1822}$$

By assuming $m = 10$, then the above Theorem in Numbers is, *viz.*

$$n = \frac{1799.725 - 10000 + 42000 + 18220}{4000 + 600n + 12600 + 1260n - 8400 - 420n - 1822}$$

$$= \frac{52019.725}{6372}.$$

That

That is.

6372) 52019.725 (8 $= n$ nearly, and then taking those Quantities where n is concerned, it becomes 11520 to be added.

$$\begin{array}{r} 6372 \\ 11520 \\ \hline 17892 \end{array}$$

17892) 52019.725 (2.9074 $= n$ more nearly.

Consequently $m+n=x=12.9074$.

Q. E. I.

(See Form 15th.)

Of

Of Surſolids, *or Roots of the* Fifth Power.

EXAMPLE 26.

GIven $x+x^2+x^3+x^4+x^5=10101010100$, or $x^5+x^4+x^3+x^2+x-10101010100=0$. } Quere x?

Aſſume $m+n=x$

Then from my Table of Powers (ſeeing the Equation is the 5th Power,) the Equation becomes, *viz.*

$$m^5+5m^4n+10m^3n^2+m^4+4m^3n+6m^2n^2+m^3+3m^2n+3mn^2+m^2+2mn+n^2+m+n=10101010100.$$

By Tranſpoſition it is.

$$5m^4n+10m^3n^2+4m^3n+6m^2n^2+3m^2n+3mn^2+2mn+n^2+n=10101010100-m^5-m^4-m^3-m^2-m.$$

Hence ariſes this General

THEOREM.

$$n=\frac{10101010100-m^5-m^4-m^3-m^2-m}{5m^4+10m^3n+4m^3+6m^2n+3m^2+3mn+2m+n+1}$$

Hence let us aſſume $m=90$, then the Theorem in Numbers will be, *viz.*

$$n=\frac{10101010100-5904900000-65610000-729000-8100-90}{328050000+7290000n+2916000+48600n+24300+270n+180+n+1}.$$

That is.

330990481) 4129762910 (12 $=n$ nearly.

Then

Then taking all the Quantities where n is concern'd, and multiply by the Value of n just now found, as viz.

$$7290000n = 87480000$$
$$48600n = 583200$$
$$270n = 3240$$
$$n = 12$$

88066452, which must be added to the first Divisor.

As 330990481
88066452
419056933) 4129762910 (9.854 = n more nearly.

Consequently $m+n = 99.853 = x = 100$ *proximè.*

Q. E. I.

(*See Form* 16*th.*)

EXAMPLE 27.

Given $-x^5+586x^4+2x^3-386808x^2+918727x = 385050$.

Assume $m+n=x$.

Then from my Table of Powers, taking the Power of the Equation, the Substitution will stand thus.

$-m^5-5m^4n-10m^3n+586m^4+2344m^3n+3516m^2n^2+2m^3+6m^2n+6mn^2-386808m^2-773616mn-386808n^2+918727m+918727n = 385050.$

By Transposition it becomes

$-5m^4n-10m^3n^2+2344m^3n+3516m^2n^2+6m^2n+6mn^2-773616mn-386808n^2+918727n = 385050+m^5-586m^4-2m^3+386808m^2-918727m.$

From which arises this

THEOREM.

$$n = \frac{385050+m^5-586m^4-2m^3+386808m^2-918727m}{-5m^4-10m^3n+2344m^3+3516m^2n+6m^2+6mn-773616m-386808n+918727}.$$

Now

Now at a venture let us assume $m = 30$, then the above Theorem in Numbers is

385050	$-586m^4 = -474660000$
$m = +24300000$	$-2m^3 = -\ 54000$
$386808m^2 = 348127200$	$-918727m = -27361810$
372812250	-502075810

Now $372812250 - 502075810 = -129263560$ for a Dividend. Here you may see that I took m too great, by Reason of its Negative Sign; then upon the Supposition of m, the Divisor being put into Numbers, it will be 38034920.

That is.

$$38034920)-129263560\ (-3 = n \text{ nearly.}$$

Then taking those Quantities in the Divisor where n is concern'd, and putting its Value just now found, we shall have -7523316, which must be taken from the Divisor, by Reason of unlike Signs; as

$$\begin{array}{l} 38034920 \\ \underline{-7523316} \\ 30511604)-129263560\ (-4.2365 = n \text{ more nearly.} \end{array}$$

Hence seeing my second Value of n to have a Negative Sign before it, shews, that I assumed m too much, and therefore must deduct n from m; as

$$m - n = x = 25.7635.$$

Q. E. I.

(See Form 16th.)

Of

Of the Square Cubed, *or* Cube Squared, *the* Sixth Power.

EXAMPLE 28.

GIVEN $x + x^2 + x^3 + x^4 + x^5 + x^6 = 100$. Quere x?

Assume as before $m + n = x$, then according to my Table of Powers, *viz.* the Sixth, the Equation becomes, *viz.*

$m^6 + 6m^5n + 15m^4n^2 + m^5 + 5m^4n + 10m^3n^2 + m^4 + 4m^3n + 6m^2n^2 + m^3 + 3m^2n + 3mn^2 + m^2 + 2mn + n^2 + m + n = 100$.

By Transposition we get

$6m^5n + 15m^4n^2 + 5m^4n + 10m^3n^2 + 4m^3n + 6m^2n^2 + 3m^2n + 3mn^2 + 2mn + n^2 + n = 100 - m^6 - m^5 - m^4 - m^3 - m^2 - m$.

Hence we get this Universal

THEOREM.

$$n = \frac{100 - m^6 - m^5 - m^4 - m^2 - m^2 - m}{6m^5 + 15m^4n + 5m^4 + 10m^3n + 4m^3 + 6m^2n + 3m^2 + 3mn + 2m + n + 1}.$$

Thence assuming $m = 1$, the Theorem reduced into Numbers, and the Operation perform'd as the Cubic Equations, or Biquadratic, &c. the Value of n will be found to be $= .67142$.

Consequently $m + n = x = 1.67142$.

Q. E. I.

(See Form 17th.)

R EXAMPLE.

EXAMPLE 29.

Given $x^6 - 1.032115x^5 - 1.467368x^4 + 1.548173x^3 + .467368x^2 - .516057x + .0665789 = 0$.

Assume $m + n = x$, then from my Table of Powers the above given Equation becomes, *viz.*

$m^6 + 6m^5n + 15m^4n^2 - 1.032115m^5 - 5.160575m^4n - 10.321150m^3n^2 - 1.467368m^4 - 5.869472m^3n - 8.804208m^2n^2 + 1.548173m^3 + 4.644519m^2n + 4.644519mn^2 + .467368m^2 + .934736mn + .467368n^2 - .516057m - .516057n + .0665789 = 0$.

By Transposition we have

$6m^5n + 15m^4n^2 - 5.160575m^4n - 10.321150m^3n^2 - 5.869472m^3n - 8.804208m^2n^2 + 4.644519m^2n + 4.644519mn^2 + .934736mn + .467368n^2 - .516057n = -.0665789 - m^6 + 1.032115m^5 + 1.467368m^4 - 1.548173m^3 - .467368m^2 + .516057m$.

Hence arises this General

THEOREM.

$$n = \frac{-.0665789 - m^6 + 1.032115m^5 + 1.467368m^4 - 1.548173m^3 - .467368m^2 + .516057m}{6m^5 + 15m^4n - 5.160575m^4 - 10.321150m^3n - 5.869472m^3 - 8.804208m^2n + 4.644519m^2 + 4.644519mn + .934736m + .467368n - .516057}.$$

Hence assuming $m = .3$, then the above Theorem put into Numbers, and the Operation had, as in our former Examples, for n, we shall get the Value of $n = .153979783$1; and consequently $m + n = .4539797831 = x$.

Q. E. I.

(*See Form* 17*th.*)

EXAMPLE

EXAMPLE 30.

Given $-x^6+4x^4+1332x^3+95x^2-3330x=443556$.

Assume $m+n=x$, then from the Table of Powers we get the following Equation, *viz.*

$-m^6-6m^5n-15m^4n^2+4m^4+16m^3n+24m^2n^2+1332m^3+3996m^2n+95m^2+190mn+95n^2-3330m-3330n=443556$.

By Transposition we get

$-6m^5n-15m^4n^2+16m^3n+24m^2n^2+3996m^2n+3996mn^2+190mn+95n^2-3330n=m^6-4m^4-1332m^3-95m^2+3330m+443556$.

Hence arises this Universal

THEOREM.

$$n=\frac{m^6-4m^4-1332m^3-95m^2+3330m+443556}{6m^5-15m^4n+16m^3+24m^2n+3996m^2+3996mn+190m+95n-3330}.$$

By assuming $m=10$, the Value of n may easily be found, which will satisfy the Conditions of the Equation.

Q. E. I.

(See Form 17th.)

I question not but by these few and choice Examples, the Nature of, and Manner how to proceed in this Method is sufficiently cleared; as to the Extraction of Roots out of simple or pure Equations, how highly soever they be.

And because there is great Care and Trouble attends the continued Involutions of $m+n$, or $m-n$, especially to any considerable Height, by Reason of the *Unciæ* (or Numeral Figures that arise by involving the Quantities) I have at the Beginning raised a Table that the Learner may have a continual Recourse to for his Operations.

Likewise, that the Products are found by making two Progressions Geometrical, the one beginning at the defired Power of the first Part of the Root, and ending at an Unit; and the other beginning at an Unit, and ending at the Power of the other Part of the Root; as if you were to find the Sixth Power of $m+n$, write the Powers thus,

m^6	m^5	m^4	m^3	m^2	m	1
1	n	n^2	n^3	n^4	n^5	n^6

$$m^6+m^5n+m^4n^2+m^3n^3+m^2n^4+mn^5+n^6$$

will be the Terms in the Sixth Power of $m+n$, by multiplying the Powers above by those below; and to find their *Unciæ*, that of the first Term is always an Unit, and that of the second is the Exponent of the first, and of the third is the Exponent of m in the second Term, multiplied by the affix'd *Unciæ*, and divided by $2=15$, and of the third is the Exponent of m in the third Term, multiplied by the prefix'd *Uncia* 15, and divided by 3, and so of the fourth, &c. which gives the Sixth Power.

$$m^6+6m^5n+15m^4n^2+20m^3n^3+15m^2n^4+6mn^5+n^6.$$

I think what has been said in this Part will be sufficient for the meanest Capacity. I shall conclude this Part by adding a few Examples, leaving them for the Learner's Perusal, by giving him the Answers only.

EXAMPLE 31.

Suppose $-x^8+1800x^6-1056272x^4+222272000x^2 = 8768000000$.

Hence by assuming $m+n$ for x as before, and ordering the Equation you will find the Value of $m\pm n=x=21.2$.

(See Forms 19th. and 20th.)

EXAMPLE 32.

$-x^8+536x^7-70350x^6+2208588x^5+141731084x^4-11101565353x^3+155776050139x^2+734886931587x-1918212972876 73=0$.

Hence

Hence $x = 63.21.$ as appears from the Table of Theorems.

EXAMPLE 33.

$$x^8 + 10.3303x^7 - 294.8875x^6 + 486515.37x^5 + 20167098.3x^4 - 270427545.014x^3 - 13736480320.5x^2 + 31359884269.94x = -229497234 8845.65.$$

Hence by our Table of Theorems we ſhall find $x = 16.04984.$

(See Forms 19th. and 20th.)

EXAMPLE 34.

Suppoſe $x^{10} - 25.6x^9 + 105.1932x^8 + 640x^7 + 4349.031x^6 - 64906.084x^4 + 18016295.945649x^2 = 150135799.54708.$

Now by aſſuming $m \pm n = x$, and ſubſtituting the Equation according to our Method we have laid down in the preceding Examples, we ſhall get the Value of $x = 4.$

I ſhall not here trouble the Reader with any more Examples of Converging Series, ſeeing I have here brought him how to ſolve any Equation whatſoever, leading him on Step by Step, till he is come to Equations of the Tenth Power I ſhall now give him a few Examples in Equations Literal, where I make a, b, c, d, &c. known Coefficients, and x, y, z, &c. unknown Quantities, or Numbers ſought; and it is from theſe Examples that I made the *Table of Converging Series*, with their Theorems for the converging n, where the Reader will meet with every Thing ſo plain, as will not admit of an Explanation, by Reaſon of its great Facility, only it muſt be obſerved.

That what Numbers are wanting in your given Equation, the ſame muſt be omitted in your Theorems; alſo Regard muſt be had to the Signs.

Of

Of CONVERGING SERIES *Literally*.

EXAMPLE I.

GIVEN $ax^3+bx^2+cx=N$. Quere x?

OPERATION.

Given $ax^3+bx^2+cx=N$.

Then $x^3+\frac{b}{a}x^2+\frac{c}{a}x=\frac{N}{a}$.

Substitute $\frac{b}{a}=p$, $\frac{c}{a}=q$, and $\frac{N}{a}=G$. Then the Equation will stand thus,

$$x^3+px^2+qx=G.$$

Assume $m+n=x$, then the Equation becomes

$$m^3+3m^2n+3mn^2+pm^2+2pmn+pn^2+qm+qn=G.$$

By Transposition.

$$3m^2n+3mn^2+2pmn+pn^2+qn=G-m^3-pm^2-qm.$$

From which arises this Universal

THEOREM.

$$n=\frac{G-m^3-pm^2-qm}{3m^2+3mn+2pm+pn+q}.$$ Q. E. I.

EXAMPLE 2.

Given $ax^3-bx^2+cx-N=0$.

SOLUTION.

$$ax^3-bx^2+cx-N=0.$$

$$x^3-\frac{b}{a}x^2+\frac{c}{a}x-\frac{N}{a}=0.$$

Substitute

Subſtitute $-\frac{b}{a}=-p$, and $\frac{c}{a}=q$, and $\frac{N}{a}=$ $-G$, then it will be $x^3-px^2+qx-G=0$.

$\pm \quad x^3-px^2+qx=G$.

Aſſume $m+n=x$, then the Equation becomes

$m^3+3m^2n+3mn^2-pm^2-2pmn-pn^2+qm+qn=G$.

By Tranſpoſition.

$3m^2n+3mn^2-2pmn-pn^2+qn=G-m^3+pm^2-qm$.

Hence ariſes this Univerſal

THEOREM.

$$n=\frac{G-m^3+pm^2-qm}{3m^2+3mn-2pm-pn+q}.$$

The Illuſtration of the above in Numbers.

Given $\overset{a}{8}x^3-\overset{b}{1440}x^2+\overset{c}{86400}x-\overset{N}{1600000}=0$.

SOLUTION.

	$8x^3-1440x^2+86400x-1600000=0$.
$\div 8$,	$x^3-180x^2+10800x-200000=0$.
$\pm$	$x^3-180x^2+10800x=200000$.

Here $-p=-180$, $10800=q$, and $200000=G$.

Then our Univerſal Theorem will be

$$n=\frac{G-m^3+180m^2-10800m}{3m^2+3mn-360m-180n+10800}.$$

Hence the Value of $m+n=x=34.7$.

W. W. R.

EXAMPLE 3.

Given $ax^3-bx^2-cx=N$.

Then

Then by Division and Substitution, as the two last Examples, the Final Equation is

$$x^3 - px^2 - qx = G.$$

Hence assuming $m + n = x$, and from my Table of Powers the Equation substituted aright, it will be

$$m^3 + 3m^2n + 3mn^2 - pm^2 - 2pmn - pn^2 - qm - qn = G.$$

Then by Transposition arises this Universal

THEOREM.

$$n = \frac{G - m^3 + pm^2 + qm}{3m^2 + 3mn - 2pm - pn - q}.$$

EXAMPLE 4.

$$ax^4 + bx^3 + cx^2 + dx = N.$$

Then dividing the Equation by the Co-efficient a, it will be

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x = \frac{N}{a}.$$

Substitute $p = \frac{b}{a}$; $q = \frac{c}{a}$; $r = \frac{d}{a}$; and $G = \frac{N}{a}$.

Then the Equation will be as follows.

$$x^4 + px^3 + qx^2 + rx = G.$$

Assuming $m + n = x$, by my Table of Powers we shall have, *viz.*

$$m^4 + 4m^3n + 6m^2n^2 + pm^3 + 3pm^2n + 3pmn^2 + qm^2 + 2qmn + qn^2 + rm + rn = G.$$

By Transposition we have

$$4m^3n + 6m^2n^2 + 3pm^2n + 3pmn^2 + 2qmn + qn^2 + rn = G - m^4 - pm^3 - qm^2 - rm.$$

THEOREM.

THEOREM.

$$n = \frac{G - m^4 - pm^3 - qm^2 - rm}{4m^3 + 6m^2n + 3pm^2 + 3pmn + 2qm + qn + r}.$$

Suppose an Example in Numbers. As

EXAMPLE 5.

$5212x^4 - 1093600x^3 + 56547625x^2 - 1585920000x + 6140484000.$

For each Co-efficient put, a, b, c, &c. respectively, and it will be

$$ax^4 - bx^3 + cx^2 - dx + N. = 0.$$

Divided, $x^4 - \frac{b}{a}x^3 + \frac{c}{a}x^2 - \frac{d}{a}x + \frac{N}{a} = 0$ in Numbers,

$$x^4 - 209x^3 + 10849x^2 - 304280x + 1178142 = 0.$$

And for each Co-efficient substitute p, q, r, &c. Then we have

$$x^4 - px^3 + qx^2 - rx + G = 0.$$

By assuming $m + n = x$, we have according to our former Operation this Universal

THEOREM.

$$n = \frac{-m^4 + pm^3 - qm^2 + rm - G}{4m^3 + 6m^2n - 3pm^2 - 3pmn + 2qm + qn - r}.$$

Whence the Reader is to observe,

That a, b, c, d, e, &c. represent the Co-efficients of the unknown Quantity x, let the Co-efficients be what they will, and N the absolute Number given.

Now seeing oftentimes that a Co-efficient is prefix'd to the highest Power of the unknown Quantity x. I have divided all the Co-efficients of the other Terms, by the Co-efficient of the highest unknown Quantity; and for their Quotients have substituted p, q, r, s, &c. and G,

 the

the absolute Number after N, was divided by a, that is, after the absolute Number given was divided by the prefix'd Co-efficient of the highest Power of the unknown Quantity x, then we have the Equation out of Algebraic Fractions, for our Operation, by substituting p, q, r, &c. aforesaid; tho' it is not always, as that the highest Power of the unknown Quantity x has a Co-efficient, then the Equation needs not Division, but putting p, q, r, &c. for b, c, d, &c. respectively: As

EXAMPLE.

Let $x^6+bx^5-cx^4+dx^3+ex^2+fx=N$, in Numbers.

Suppose $x^6+1000x^5-200x^4+100x^3+50x^2+40x=500000$.

Now by substituting $p=1000$; $-200=-q$; $100=r$; $50=s$; $40=t$; $500000=G$; then the above Equation becomes

$$x^6+px^5-qx^4+rx^3+sx^2+tx=G.$$

Here you see, that p, q, r, &c. represent b, c, d, &c. respectively, because the Equation, is not reduced any lower, but the final Equation just the same as the given one. But

Suppose $ax^6+bx^5-cx^4+dx^3+ex^2+fx=N$. Let the same in Numbers be

$$20x^6+800x^5-700x^4+40x^3+20x^2+500x=2000000.$$

Here you see is a Co-efficient (*viz.* 20) prefix'd before the highest Power of the unknown Quantity x, (*viz.* x^6) which must be divided off, that is, I must divide all the other Co-efficients by 20, and then arises this Equation.

$$x^6+40x^5-35x^4+2x^3+x^2+25x=100000.$$

Thence substituting p, q, r, &c. for the Co-efficients, there arises this final Equation.

$$x^6+px^5-qx^4+rx^3+sx^2+tx=G.$$

Hence

Hence the Equation is reduced for Converging Series, and may be ſolv'd by our Univerſal Theorems. Which laſt Equation I call final, by Reaſon that now I can, by aſſuming $m \pm n = x$ (which m is = to the known Part of the Root ſought x, and muſt be taken as near the Root as may be, whether it be greater or leſs than the ſaid Root, and n is = the unknown Part of the Root ſought, whoſe Value may be Negative or Affirmative, according as that of m is taken greater, or leſs than the Truth) find the Value of x, and it is from theſe Principles I have raiſed the following Theorems, which will ſerve for any Equation of the ſame kind, by only obſerving the Signs. It is here obſerved, that when there is no Co-efficient prefix'd to the higheſt Power of the unknown Quantity x, then N and G are equal to each other reſpectively. For when any Term is wanting in the Equation, the ſame muſt be omitted in the Theorem.

A more GENERAL METHOD *for* CONVERGING SERIES.

WHICH was communicated to me by a Member of the Royal Society, for whoſe Name I have the greateſt Veneration, and ſhould have informed the World, from whom I received ſuch a Favour, had he not deſired the contrary.

Let N = Abſolute Number in any Equation.
n = Exponent of the higher Power.
x = Root or Quantity ſought.
m = any known Number taken at Pleaſure.
n = an unknown Number.
$1.\ p.\ q.\ r.\ s.$ &c. = to the reſpective Co-efficients of the given Equation; then will $m+n=x$, and

$$\mp 1 \times x^{n} \mp p \times x^{n-1} \mp q \times x^{n-2} \mp r \times x^{n-3} \mp, \&c. = N,$$

repreſent any Equation whatſoever; and becauſe $m+n = x$, ſuch a General Equation may be thus expreſſed.

$\mp 1 \times \overline{m+n}|^{n} \mp p \times \overline{m+n}|^{n-1} \mp q \times \overline{m+n}|^{n-2} \mp r \times \overline{m+n}|^{n-3} \mp = N.$

But to bring this to Converging Series, it is first necessary to prove, that every Power raised from a Binomial (without regarding the Co-efficients) consists, or is composed of two Ranks or *Series* of Powers, one increasing from n^{n-n}, or 1 to n^{n}, and the other decreasing from m^{n} to m^{n-1}, or 1; and each Member in one is multiplied into its corresponding Member, in the other respectively, as may appear thus.

$$\overline{m+n}|^{2} \left\{ \begin{matrix} mm \\ 2mn \\ nn \end{matrix} \right\} = \left\{ \begin{matrix} m^{n} \times n^{n-n}, \text{ or } 1. \\ m^{n-1} \times n^{n-1} \text{ twice}. \\ m^{n-n} \times n^{n}. \end{matrix} \right.$$

Again.

$$\overline{m+n}|^{2} \left\{ \begin{matrix} mmm \\ 3mmn \\ 3mnn \\ nnn \end{matrix} \right\} = \left\{ \begin{matrix} m^{n} \times n^{n-n}. \\ m^{n-3} \times n^{n-2} \\ m^{n-2} \times n^{n-1} \\ m^{n-n} \times n^{n} \end{matrix} \right\} \text{thrice}.$$

And so it will be *ad Infinitum*.

Q. E. D.

Hence these two Corollaries.

Corollary 1.

That the Co-efficient of the second Term in any Power raised from a Binomial is always $= n$, the Exponent of the highest Power.

Corollary 2.

That the Root or Side of n^{n}, the unknown Quantity, is always multiplied into the Second Term of the known.

Now

Now from the Latter, it is evident we are (in this Case, but to make use of the two Members of the Power of such *Binomial*, and by the first we may express the Co-efficient of the second Term by n, the Exponent of the Power: Therefore the former Equation will now stand thus.

$$\pm 1 \times m^{n} \pm n\, \overline{{}_{n}m^{n-1}\, n} \pm p \times m^{n-1} \pm \overline{{}_{n-1}m^{n-2}\, m} \pm q \times m^{n-2} \pm {}_{n-2}m^{n-2}\, n \pm r \times m^{n-3} \pm {}_{n-3}m^{n-4}\, n \pm \&c. = N.$$

Now to find the Value of n, or the unknown Quantity: It is plain, that those Members into which it is multiplied will be the Divisor with the same Signs, as being to be transposed to the other Side of the Equation. Therefore first we get the following

THEOREM.

$$n = \frac{N \pm 1 \times m^{n} \mp p \times m^{n-1} \pm q \times m^{n-2} \pm r \times m^{n-3}, \&c.}{1 \times {}_{n}m^{n-1} \pm p \times {}_{n-1}m^{n-2} \pm q \times {}_{n-2}m^{n-3} \pm r \times {}_{n-3}m^{n-4}, \&c.}$$

Which Theorem exhibits all possible particular ones, for extracting of Roots according to the first Sort of Mr. *Ralphson*'s, agreeing exactly with them, as will be found on Trial, always remembring that the Signs in the *Dividend* must be contrary to those in the Equation, and in the Divisor the same respectively.

But $m + n = x$. Therefore Secondly,

THEOREM.

THEOREM.

$$x = \frac{N \pm 1 \times {}_{n-1}m^{n} \pm p \times {}_{n-2}m^{n-1} \pm q \times {}_{n-3}m^{n-2} \pm r \times {}_{n-3}m^{n-3}, \&c.}{1 \times m^{n-1} \pm p \times {}_{n-1}m^{n-2} \pm q \times {}_{n-2}m^{n-3} \pm r \times {}_{n-3}m^{n-4}, \&c.}$$

Which gives all thoſe of the ſecond Sort univerſally.

But in this Caſe the Signs, both in the Dividend and Diviſor, will be the ſame, as in the given Equation reſpectively; as likewiſe it may be proper to take Notice, *That if any Term be wanting in the Equation, the ſame muſt be omitted in either Theorem reſpectively.*

Now from either of theſe *two Generals*, to deduce any particular Theorem for finding the Root of any given Equation, we need only conſider, that $m^{n-n} = 1$. or $\frac{m^n}{m^n} = 1$. that Unity will neither multiply nor divide; alſo that $n - n^m = 0$, or $n^m - n^m = 0$; and any Quantity multiplied into Nothing is $= 0$, and when either Caſe happens (which always will, except where the laſt Term is wanting, the Theorem is determined.

THEREFORE

THEREFORE.

Suppose an Equat. as		By the 1st Gen. Theor.		By the 2d Gen. Theor.
$x^2 = N$	Then we shall have $x =$	$\frac{N - m^2}{2m}$	$x =$	$\frac{N + m^2}{2m}$
$x^3 = N$		$\frac{N - m^3}{3m^2}$		$\frac{N + 2m^3}{3m^2}$
$x^4 = N$		$\frac{N - m^4}{4m^3}$		$\frac{N + 3m^4}{4m^3}$
$x^2 \pm px = N$		$\frac{N - m^2 \mp pm}{2m \pm p}$		$\frac{N + m^2}{2m \pm p}$
$-x^2 \pm px = N$		$\frac{N + m^2 - pm}{-2m + p}$		$\frac{N - m^2}{-2m + p}$
$x^4 \pm px^3 = N$		$\frac{N - m^4 + pm^3}{4m^3 \pm 3pm^2}$		$\frac{N + 3m^4 \pm 2pm^3}{4m^3 \pm 3pm^2}$
$x^3 \pm px^2 \pm qx = N$		$\frac{N - m^3 \mp pm^2 \mp qm}{3m^2 \pm 2pm \pm q}$		$\frac{N + 2m^3 + pm^2}{3m^2 \pm 2pm \pm q}$

After the same Manner for any Equation whatsoever.

Thus having the particular *Theorem*, the *Application* in either Case is as follows.

Let m be any Number taken at Pleasure as before.

T = Theorem, in which m must be of its last Value found.

Then the Process will be of the

First General Theorem.	Second General Theorem.
m the 1st. $\pm$ T $= m$ the 2d. Then	T $= m$ the 2d. Then
m the 2d. $\pm$ T $= m$ the 3d. Then	T $= m$ the 3d. Then
m the 3d. $\pm$ T $= m$ the 4th. Then	T $= m$ the 4th. Then
m the 4th. $\pm$ T $= m$ the 5th. &c.	T $= m$ the 5th. &c.

Some of which true Values of m will terminate in the true Root sought, if it have one: But if it be a Surd, then the Value of m will proceed into an *Infinite Series*, but may be prosecuted nearer the Truth than any assignable, which *Series*, each *Operation*, will proceed in *Number of Places*, in a *Geometrical Progression*, whose first Term

Term is 1, and Ratio = 2, *viz.* First 1, then 2, then 4, then 8, then 16, then 32, then 64. &c. Places.

It is likewise observable, that the first *General Theorem* converges by finding out a Number to be added to, or subtracted from the *last* Value of m (as it shall be adfected with $+$ or $-$) untill m be $= x$ sought. So the last converges by m itself, whose Value, at each Operation, shall grow nearer and nearer, untill it be $= x$ sought.

We may also take Notice, that tho' m be assumed never so far from the Root, yet it will converge to it by renewing the Operation.

But the Work may be much shortened, in Case we point the given Equation (if it will admit of it) both in the *absolute Number* and *Co-efficient*, according to their respective Degrees of *Adfection*; and take first 1. then 2, then 4, &c. of those Points (from the first) each Operation: For it is evident, the Co-efficients increase their Powers, as the highest known Term decreases; therefore the absolute Number is of the same Power, with the highest unknown Quantity.

One Instance may be sufficient to explain it. Suppose this Cubic Equation to be pointed, *viz.*

$$x^3 + 25x^2 + 836x = 53297.$$
$$\text{or } x^3 + px^2 + qx = \text{N}.$$

Then it would be $x^3 + \dot{2}\dot{5}x^2 + \dot{8}3\dot{6}x = \dot{5}3\dot{2}9\dot{7}.$

For the absolute Number is a Cube }
Co-efficients q a Square } and are pointed accordingly.
p a Lateral }

And the like Method for any other Equation, where it will admit of it.

Now to apply this we are to take

The first Operation $x^3 + 2x^2 + 8x = 53.$

Second Operation $x^3 + 25x^2 + 836x = 53297$, and consequently the Value of the Co-efficients, as well as the absolute Numbers alters, so long as there are Punctations.

But

But by a Numerical Operation, the ſaid Notification, as well as the Method of the Proceſs of each Theorem, will be further illuſtrated. Therefore,

1. Suppoſe $x^2 = 2 = \text{N}$, ſeek x by the firſt General Theorem.

Then $n = \frac{\text{N}-\text{N}^2}{2m} = \text{T}$, and take $m = 1$.

Therefore $1+\text{T}\ (=.5) = 1.5 = m$ the 2d.

∵ $1.5-\text{T}\ (=-.088) = 1.417 = m$ the 3d.

∵ $1.417-\text{T}(=-.002783)=1.414217=m$ the 4th.

∵ $1.414217 - \text{T}\ (= -.000003437622) = 1.414213562378 = m$ the 5th $= x$.

2. Suppoſe $xx = 2 = \text{N}$. ſeek x by the ſecond General Theorem.

Then $x = \frac{\text{N}+\text{N}^2}{2m} = \text{T}$, and take $m=1$, as before.

Therefore $\text{T} = 1.5 = m$ the 2d.

∵ $\text{T} = 1.416 = m$ the 3d.

∵ $\text{T} = 1.414215 = m$ the 4th.

∵ $\text{T} = 1.41421356237 3=m$ the 5th. $= x$.

By which it is evident, Firſt, that both Theorems amount to the ſame Thing, the Difference being only in the laſt Figure, which would be corrected the next Operation. Secondly, that x will proceed into an *Infinite Series*, if a Surd. Thirdly, that each Operation gives double the Number of Figures of the laſt.

3. Suppoſe $x^4=28398241=\text{N}$. ſeek x by Theorem 1.

Then $n = \frac{\text{N}-m^4}{3m^3} = \text{T}$, and take $m = 10$.

Therefore $10-\text{T}\ (=-3) = 7 = m$ the 2d.

∵ $7+\text{T}\ (=+.4)=7.4=m$ the 3d.

∵ $7-\text{T}\ (=-.1)=7.3=x$, the true Biquadratic Root ſought.

4. Suppoſe $x^4= 2839.8241 = \text{N}$. as before, ſeek x by the ſecond Theorem.

T Then

Then $x = \frac{N - 3m^4}{4m^3} = T$, and take $m = 5$.

Therefore $T = 5.6 = m$ the 2d.

∵ $T = 8.2 = m$ the 3d.

∵ $T = 7.4 = m$ the 4th.

∵ $T = 7.3 = m$ the 5th. $= x =$ to the true Root fought.

From which two laft Examples it appears, Firft, that either Theorem will find the true Root, if it have one. Secondly, that it matters not, whether m be taken above or below the Root, or how far from it.

5. Suppofe $x^2 + \dot{5}\dot{8}\dot{7}x = \dot{9}\dot{8}\dot{7}\dot{4}59$, or $xx + px = N$. Seek x by Theorem the firft (*i. e.*) $n = \frac{N - m^2 - pm}{2m + p} = T$, becaufe of the Punctations we are to take.

1. Operation $xx + 5x = 98$
2. - - - $xx + 58x = 9874$
3. - - - $xx + 587x = 987459$

} and fuppofe $m = 8$.

Therefore $8 - T(= -.2) = 78 = m$ the 2d.

∵ $78 - T(= -3.4) = 746 = m$ the 3d.

∵ $746 - T(= -3.34) = 742.66 = m$ the 4th.

∵ $742.66 - T(= -.012689) = 742.647311 = x$ fought.

Again.

6. Suppofe $xx - 20x = 53482$, or $xx - px = N$. Seek x by the fecond General Theorem.

Then $x = \frac{N + m^2}{2m - p} = T$, and take $m = 250$.

Therefore $T = 241 = m$ the 2d.

∵ $T = 241.4 = m$ the 3d.

∵ $T = 241.475 = m$ the 4th.

∵ $T = 241.477860 = m$ the 5th. $= x$ fought.

From thefe two laft it is plain; Firft, that there is no abfolute Neceffity for Punctation.

Secondly,

Secondly, That Punctation does nevertheless *shorten* the Work where it can be done.

But I hope I have said enough to make the whole Matter, as well as the Manner of proceeding plain and easy to the meanest Capacity; and though I have given Numerical Examples, no farther than an affected Quadratic, yet it is the same to any Degree of Power, or Affectation whatsoever, Regard being had to its proper and particular Theorem deduc'd from either of the General Ones. I shall therefore now proceed to Roots in General; but first I will shew by an Example how the Cube may be compleated as a Quadratic.

A new Way of Compleating the Cube.

SUppose $x^3+12x^2+48x=152$. Here you see it is a perfect Cubic Equation: Now the same may be compleated thus.

I observed the Canon for the Cube Root ($x^3+3bx^2+3b^2x+b^3$) and I found the third of the Co-efficient (or $3b$) cub'd, is always the fourth Term of a regular Cube, which I tried in this Case, calling $12=3b$, and $48=3b^2$, the Equation then will stand thus, $x^3+3bx^2+3b^2x=152$; then adding 64 the Cube of 6, a third of the Co-efficient 36, to both Sides of the Equation, and we shall have $x^3+3bx^2+3b^2x+64=(152+64)\ 216$, and extracting the Root $x+4=\sqrt[3]{216}=6$. and $x=2$.

OPERATION.

Equation,	1	$x^3+12x^2+48x=152$
per Canon,	2	$x^3+3bx^2+3b^2x=152$
2 c □	3	$x^3+3bx^2+3b^2x+64=216$.
3 ω 3	4	$x+4=\overline{216}\rvert^{\frac{1}{3}}=6$
4—4	5	$x=6-4=2$.

Q. E. D.

 An

An Universal Solution *of* Cubic *and* Biquadratic Equations, *analytically*.

§. 1. *Of the Universal* Cubic Equation.

$$x^3 = 3px^2 + 3qx + 2r.$$
$$\qquad -3p^2 + p^3.$$
$$\qquad\qquad -3pq.$$

There are three Roots, *viz.*

1. $x = p + \overline{r + \sqrt{r^2 - q^3}}\,\big|^{\frac{1}{3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}.$

2. $x = p - \frac{1 - \sqrt{-3}}{2} \times \overline{r + \sqrt{r^2 - q^3}}\,\big|^{\frac{1}{3}} - \frac{1 + \sqrt{-3}}{2} \times \overline{r - \sqrt{r^2 - q^3}}\,\big|^{\frac{1}{3}}.$

3. $x = p - \frac{1 + \sqrt{-3}}{2} \times \overline{r + \sqrt{r^2 - q^3}}\,\big|^{\frac{1}{3}} - \frac{1 - \sqrt{-3}}{2} \times \overline{r - \sqrt{r^2 - q^3}}\,\big|^{\frac{1}{3}}.$

That the Arithmetical *Calculus* may appear the eaſier, and fitter for Operation, put the Cube Root of the irrational Binomial $r + \sqrt{r^2 - q^3}$ to be $m + \sqrt{n}$, the three Roots of the ſame Equation will be $x = p + 2m$, and $x = p - m \pm \sqrt{-3n}$.

Therefore in any given Cubic Equation, we muſt compare p, q, r, &c. for theſe being known, all the Roots of the Equation become known.

Let the Root x be ſought of this Cubic Equation, *viz.*

§. 1. $x^3 = 2x^2 + 3x + 4.$

Hence

Hence it will be according to the Prescript, $3p=2$, or $p=\frac{2}{3}$. Secondly, $3q-(3p^2)\frac{4}{3}=3$. or $q=\frac{13}{9}$. Thirdly, $2r\ (+\overline{p^2-3q}\times p)-\frac{70}{27}=4$, or $r=\frac{89}{27}$, and $r^2-q^3=\frac{212}{27}$. And therefore $x=\frac{2}{3}+\sqrt[3]{\frac{89}{27}+\sqrt{\frac{212}{27}}}+\sqrt[3]{\frac{89}{27}-\sqrt{\frac{2}{27}}}$, the other Roots are impossible.

§. 2. In this Equation $x^3=12x^2-41x+42$: In the first Place it will be $3p=12$, or $p=\frac{12}{3}=4$. Secondly, $3q-(3p^2)48=-41$, or $q=\frac{7}{3}$. Thirdly, $2r+(\overline{p^2-3q}\times p)36=42$, or $r=3$. and from thence $r^2-q^3=-\frac{100}{27}$. But the Cube Root of the Binomial Surd, $3+\sqrt{-\frac{100}{27}}\ (=r+\sqrt{r^2-q^3})$ is to be extracted according to the fought Methods of Arithmetic, and is $-1+\sqrt{-\frac{4}{3}}$, $(=m+\sqrt{n},)$ and therefore the Root $x=(p+2m=4-2=)\ 2$; or likewise $x=(p-m\pm\sqrt{-3n}=4+1\pm(\sqrt{4})\ 2=)\ 7$ or 3. Again, the other Root of the same Binomial $3+\sqrt{-\frac{100}{27}}$ is $\frac{3}{2}+\sqrt{-\frac{1}{12}}$ $(=m+\sqrt{n})$ and therefore the Root $x=(p+2m=4+3=)\ 7$; and likewise $x=(p-m\pm\sqrt{-3n}=4-\frac{3}{2}\pm\left(\sqrt{\frac{1}{4}}\right)\frac{1}{2}=3$ or 2. And again,

The third Root of the same Binomial, *viz.* $3+\sqrt{-\frac{100}{27}}$ is $-\frac{1}{2}-\sqrt{-\frac{25}{12}}$ $(=m+\sqrt{n},)$ and therefore

fore the Root $x = (p + 2m = 4 - 1 =)\ 3$, and alſo $x = (p - m \pm \sqrt{-3n} = 4 + \frac{1}{2} \pm (\sqrt{\frac{25}{4}})\frac{5}{2} =)\ 7$ or 2.

§ 3. Given in this Equation $x^3 = -15x^2 - 84x + 100$, p will be $= -5$; $q = -3$; or $r = 135$, and the Root of the Binomial $135 + \sqrt{18252}$ is $3 + \sqrt{12}$: Therefore the Root x is $= -5 + 6 = 1$, and $x = -5 - 3 \pm \sqrt{-36} = -8 + \sqrt{-36}$, impoſſible.

Again.

§ 4. Given $x^3 = 34x^2 - 310x + 1012$; here p will be $= \frac{34}{3}$, $q = \frac{226}{9}$, $r = \frac{5536}{27}$, and the Root of the Binomial $\frac{5536}{27} + \sqrt[2]{\frac{707560}{27}}$ is $\frac{16}{3} + \sqrt{\frac{10}{3}}$. Therefore the Root $x = \frac{34}{3} + \frac{32}{3} = 22$; and $x = \frac{34}{3} - \frac{16}{3} \pm \sqrt{-10} = 6 \pm \sqrt{-10}$. impoſſible.

Again.

§ 5. In the Equation $x^3 = 28x^2 + 61x - 4048$, p will be here $= \frac{28}{3}$, $q = \frac{967}{9}$, $r = -\frac{25010}{27}$, and the Root of the Binomial $-\frac{25010}{27} + \sqrt{-382347}$ is $= \frac{41}{6} + \sqrt{-\frac{243}{4}}$. Therefore $x = \frac{28}{3} + \frac{41}{3} = 23$, and $x = \frac{28}{3} - \frac{41}{6} \pm (\sqrt{\frac{729}{4}})\frac{27}{2} = 16$, or -11.

Again.

§ 6. In this Equation $x^3 = -x^2 + 166x - 660$; here p will be $= -\frac{1}{3}$; $q = \frac{499}{9}$; $r = -\frac{9658}{27}$, and the

the Root of the Binomial $-\frac{9658}{27}+\sqrt{-\frac{1147205}{27}}$ is $-\frac{22}{3}+\sqrt{-\frac{5}{3}}$. Therefore $x=-\frac{1}{3}-\frac{44}{3}=-15$, and $x=-\frac{1}{3}+\frac{22}{3}\pm\sqrt{5}=7\pm\sqrt{5}$ irrational.

Again.

§ 7. In this Equation $x^3=63x^2+99673x+9951705$, $p=21$, $q=\frac{100996}{3}$, $r=6031680$, and the Root of the Binomial $6031680+\sqrt{-\frac{47887175043136}{27}}$ is $= 183+\sqrt{-\frac{529}{3}}$. Therefore $x=21+336=387$; and $x=21-183\pm(\sqrt{529})\,23=-139$, or 185.

And after the same Manner must we proceed with the rest. And here a Theorem is investigated after the following Manner; I put the Root z of any Cubic Equation $=a+b$, and by raising the said $a+b$ to a Cube, it will be $z^3=(a^3+3a^2b+3ab^2+b^3=)\ a^3+3ab\times\overline{a+b}+b^3$. Now in the Place of $a+b$, substitute its Value, and it will be $z^3=3abz+a^3+b^3$, which Equation is constructed from the Root $z=a+b$, which Equation wants the second Term. But as this may appear more evident according to our *Formula*, I take the Equation $z^3=3qz+2r$, which transforms itself into $z^3=3abz+a^3+b^3$, and by transforming of this, it will be in the first Place $3q=3ab$, or $q^3=a^3b^3$; and secondly, $2r=a^3+b^3$, or $2ra^3=(a^6+a^3b^3=)\ a^6+q^3$, and this Quadratic Equation solv'd will be $a^3=r+\sqrt{r^2-q^3}$, and $b^3=(2r-a^3=)\ r-\sqrt{r^2-q^3}$, and therefore $a=\overline{r+\sqrt{r^2-q^3}}|^{\frac{1}{3}}$ and $b=\overline{r-\sqrt{r^2-q^3}}|^{\frac{1}{3}}$, and therefore in this Equation

$z^3=$

$z^3 = 3qz + 2r$, the Root will be $z = (a + b =)$ $\overline{r^2 + \sqrt{r^2 - q^3}})^{\frac{1}{3}} + \sqrt[3]{r^2 - r^2 - q^3}$.

But the Root is threefold, and may be changed by a threefold Value, and $\overline{r + \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$, and $\overline{r - \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$, for the Cube Root of any Quantity will be threefold, and the Root of Unity is either 1, or $-\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, or $-\frac{1}{2} - \frac{1}{2}\sqrt{-3}$.

Therefore if $1 \times \overline{r + \sqrt{r^2 - q^3}})^{\frac{1}{3}}$, or $\overline{r + \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$, $= \overline{1 \times r + \sqrt{r^2 - q^3}}|^{\frac{1}{3}} = \sqrt[3]{1} \times \overline{r + \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$, it shews some Root (as above-named, *viz.* $m + \sqrt{n}$, or $1 \times \overline{m + \sqrt{n}}$,) of the Cube $r + \sqrt{r^2 - q^3}$; now $\frac{-1 + \sqrt{-3}}{2} \times \overline{r + \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$, and $\frac{-1 - \sqrt{-3}}{2} \times \overline{r + \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$ [that is $\frac{-1 + \sqrt{-3}}{2} \times \overline{m + \sqrt{n}}$, and $\frac{-1 - \sqrt{-3}}{2} \times \overline{m + \sqrt{n}}$] will shew two other Roots of the same Cube.

And likewise $\overline{r - \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$, $\frac{-1 + \sqrt{-3}}{2} \times \overline{r - \sqrt{r^2 - q^3}}|^{\frac{1}{3}}$, and $\frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$ [that is $\overline{m - \sqrt{n}}$, $\frac{-1 + \sqrt{-3}}{2} \times \overline{m - \sqrt{n}}$, $\frac{-1 - \sqrt{-3}}{2} \times \overline{m - \sqrt{n}}$] will be three Roots of the Apotome, $r - \sqrt{r^2 - q^3}$; and by duly connecting these Roots, z will become $= \sqrt[3]{r + \sqrt{r^2 - q^3}} + \sqrt[3]{r - \sqrt{r^2 - q^3}}$ [that is, $z = \overline{m + \sqrt{n}} + \overline{m - \sqrt{n}} = 2m$,] $z = \frac{-1 + \sqrt{-3}}{2} \times \sqrt[3]{r + \sqrt{r^2 - q^3}} + \frac{-1 - \sqrt{-3}}{2} \times \sqrt[3]{r - \sqrt{r^2 - q^3}}$, [*i. e.*

$z =$

$$z = \frac{-1+\sqrt{-3}}{2} \times \overline{m+\sqrt{n}} + \frac{-1-\sqrt{-3}}{2} \times \overline{m-\sqrt{n}}$$

$= -m + \sqrt{-3n}$] and $z = \frac{-1-\sqrt{-3}}{2} \times \sqrt[3]{r+\sqrt{r^2-q^3}}$

$+ \frac{-1+\sqrt{-3}}{2} \times \sqrt[3]{r-\sqrt{r^2-q^3}}$ [*i. e.* $z = \frac{-1-\sqrt{-3}}{2}$

$\times \overline{m+\sqrt{n}} + \frac{-1+\sqrt{-3}}{2} \times \overline{m-\sqrt{n}} = -m-\sqrt{-3n}$,]

which will be the three Roots of the Equation $z^3 = 3qz + 2r$.

Now these Roots duly connected according to the preceding Method (which so connected, and by the common Method continually brought one into another) make the Equation $x^3 = 3qz + 2r$. Lastly, make $z = x - p$, and $x^3 - 3px^2 + 3p^2x - p^3 = 3qx - 3pq + 2r$ universally, the Roots of which appear, as they have been exhibited above.

It is here to be noted, that all the Roots of every Cubic Equation are possible and real, as oft as the irrational Member of the Binomial $\sqrt{r^2-q^3}$ contains the Impossibility in itself; that is, as oft as q is an Affirmative Quantity, and its Cube likewise greater than the Square by the Letter r.

But if this Member $\sqrt{r^2-q^3}$ be possible, that is, if q be a Negative Quantity; or likewise if the Cube Affirmative be less than the Square by the Letter r, then the Equation hath only one possible and real Root, and the other two are impossible.

In this Theorem, if p be made $= 0$, that is, if the second Term be wanting, then come we to *Cardan's* Method, whose Solution is shewn in the preceding.

§ 2. *Of* Biquadratic Equations, *universally*.

$$\begin{array}{l} x^4 = 4px^3 + 2qx^2 + 8rx + 4s. \\ \qquad\qquad\quad -4p^2 \;\, -4pq - q^2. \end{array}$$

The four Roots are $x = p - a \pm \sqrt{p^2 + q - a^2 - \frac{2r}{a}}$,

and $x = p + a \pm \sqrt{p^2 + q - a^2 + \frac{2r}{a}}$, where a^2 is the Root of the Cubic Equation.

$$\begin{array}{l} a^6 = p^2a^4 - 2pra^2 + r^2. \\ \quad\;\; + q \qquad\quad\;\; - s. \end{array}$$

Now in any given Biquadratic Equation, Regard is to be had in all the Terms of this Universal Equation, how p, q, r, s, will the soonest be found, and these being known, the Value of a will be found from the above Theorem; and then lastly all the given Roots of the Equation become known.

Take an Example or two for its Illustration.

EXAMPLE.

Let it be required to extract the Root of this Biquadratic Equation, *viz.* $x^4 = 8x^3 + 83x^2 - 162x - 936$. According to the Prescript it will be, first, $4p = 8$, or $p = 2$. Secondly, $2q - (4p^2)\ 16 = 83$, or $q = \frac{99}{2}$. Thirdly, $8r - (4pq)\ 396 = -162$, or $r = \frac{117}{4}$. Fourthly, $4s - (q^2)\ \frac{9801}{4} = -936$, or $s = \frac{6057}{16}$; from hence

$p^2 + q$

$p^2+q=\frac{107}{2}$, $2pr+s=\frac{7929}{16}$, $r^2=\frac{13689}{16}$, and therefore $a^6=\frac{107}{2}a^4-\frac{7929}{16}a^2+\frac{13689}{16}$.

Now as this Cubic Equation may be resolved into its Roots, we must have Recourse to the preceding Theorem, in which p will be $=\frac{107}{2}$; $q=\frac{22009}{144}$; $r=\frac{2903923}{1728}$; and $r^2-q^3=-\frac{11940075}{16}$; Consequently the Cube Root of the Binomial $\frac{2903923}{1728}+\sqrt{-\frac{11940075}{16}}$ is $-\frac{53}{12}+\sqrt{-\frac{400}{3}}$; and therefore $a^2=\frac{107}{6}-\frac{53}{6}$ $=9$; and likewise $a^2=\frac{107}{6}+\frac{53}{12}\pm(\sqrt{400})\ 20$ $=\frac{169}{4}$, or $\frac{9}{4}$. Moreover,

There are six Roots of the above Cubo-Cubic Equation, *viz.* $a=\pm 3$; $a=\pm\frac{13}{2}$, and $a=\pm\frac{3}{2}$, any one of which will indifferently serve our Purpose.

Suppose in the present Case $a=3$; x will be according to the Theorem $=(p-a\pm\sqrt{p^2+q-a^2-\frac{2r}{a}}$ $=2-3\pm\sqrt{4+\frac{99}{2}-9-\frac{39}{2}}=-1\pm(\sqrt{25})\ 5=)$ 4, or -6, and $x=(p+a\pm\sqrt{p^2+q-a^2+\frac{2r}{a}}=2$ $+3\pm\sqrt{4+\frac{99}{2}-9+\frac{39}{2}}=5\pm(\sqrt{64})\ 8=)\ 13$, or -3, which are the four Roots of the given Equation.

2. In this Equation $x^4=20x^3+252x^2-6592x+21312$, p will be $=5$; $q=176$; $r=-384$; and $s=13072$: From hence $p^2+q=201$, $2pr+s=9232$, and $r^2=147456$; and from thence $a^6=201a^4-9232a^2+147456$.

147456.

147456. Now in our Theorem for Cubics p will be $= 67$; $q = \frac{4235}{3}$, and $r = 65219$: The Cube Root of the Binomial $65219 + \sqrt{\frac{388889307072}{27}}$ will be $\frac{77}{2} + \sqrt{\frac{847}{12}}$. Therefore $a^2 = 67 + 77 = 144$, or $a = 12$; and therefore $x = 5 - 12 \pm \sqrt{23 + 176 - 144 + 64} = -7 + (\sqrt{121})\ 11 = 4$, or -18, and $x = 5 + 12 \pm \sqrt{25 + 176 - 144 - 64} = 17 \pm \sqrt{-7}$, impossible.

But the Invention of this *Theorem* is such; of the Multiplication of two Quadratic Equations $z^2 + 2az - b = 0$; and $z^2 - 2az - c = 0$, into one another, I make the Biquadratic Equation $z^4 = \overline{4a^2 + b + c} \times z^2 + \overline{2ac - 2ab} \times z - bc$, whose second Term is wanting, which I equal in Value, this Equation $z^4 = ez^2 + fz + g$. From whence, First, $4a^2 + b + c = e$, or $b = e - 4a^2 - c$. Secondly, $2ac - 2ab = 4$, that is, $2ac - 2ae + 8a^3 + 2ac = f$; or $c = \frac{f}{4a} + \frac{e}{2} - 2a^2$, and from thence $b = (e - 4a^2 - c =) - \frac{f}{4a} + \frac{e}{2} - 2a^2$. Thirdly, $-bc = g$, or $-\frac{f^2}{16a^2} + \frac{e^2}{4} - 2ea^2 + 4a^4 = -g$, that is, $a^6 = \frac{1}{2}ea^4 - \frac{1}{4}ga^4 - \frac{1}{16}ea^2 + \frac{f^2}{64}$, which Equation, as if Cubic, of the Root a^2, ef being known or taken so, is produced g; and therefore this Root may be had by the above *Theorem*, and b and c become known by the same Calculus. But the Roots of the Equations $z^2 + 2az - b = 0$, and $z^2 - 2az - c = 0$, are $z = -a \pm \sqrt{a^2 + b}$, and $z = a \pm \sqrt{a^2 + c}$, or $z = -a \pm \sqrt{\frac{1}{2}e - a^2 - 4a}$, and $z = a \pm \sqrt{\frac{1}{2}e - a^2} + \frac{f}{4a}$. Moreover, the Roots of the Equation z^4 will be $= ez^2 + fz + g$; *viz.* a or a^2

of

of the Equation a^6 is known to be $= \frac{1}{2} ea^4 - \frac{1}{4} ga^2 - \frac{1}{16} ea + \frac{f^2}{64}$. Now that this Equation may appear Univerfal, and be compleated with all its Terms, make $z = x - p$, and $x^4 - 4px^3 + 6p^2x^2 - 4p^3x + p^4$ will be $= ex^2 - 2pex + p^2e + fx - fp + g$; fo alfo $x = p - a \pm \sqrt{\frac{1}{2}e - a^2 - \frac{f}{4a}}$, and $x = p + a \pm \sqrt{\frac{1}{2}e - a^2 + \frac{f}{4a}}$. Now for Brevity and Elegancy, make $e = 2q + 2p^2$, and $f = 8r$; then $x^4 - 4px^3 + 4p^2x^2 = 2qx^2 - 4pqx + 2p^2q + p^4 + 8rx - 8pr + g$, $x = p - a \pm \sqrt{p^2 + q - a^2 - \frac{2r}{a}}$; $x = p + a \pm \sqrt{p^2 + q - a^2 + \frac{2r}{a}}$, and $a^6 = \overline{p^2 + q} \times \overline{a^4 - \frac{1}{4}g + \frac{1}{4}p^4 + \frac{1}{2}p^2q + \frac{1}{4}q^2} \times \overline{q^2 + r^2}$. Laftly, make $g = 4s - q^2 + 8pr - p^4 - 2p^2q$. Thence are made the preceding Equations.

$$\begin{array}{l} x^4 = 4px^3 + 2qx^2 + 8rx + 4s. \\ \qquad\qquad\quad - 4p^2 - 4pq - q^2. \end{array}$$

$$\begin{array}{l} \text{and } a^6 = p^2a^4 - 2pra^2 + r^2. \\ \qquad\quad + q \qquad\quad - s. \end{array}$$

That is they all appear as plac'd above.

Q. E. D.

An

An ANALYTICAL SOLUTION *of certain infinitesimal Equations, translated out of the* Latin, *from the* Philosophical Transactions *of the acute Mathematician Mr.* ABRAHAM DE MOIVRE, F. R. S.

LET n be any Number, x the unknown Quantity, or the sought Root of the Equation, and let a be any Quantity likewise known, or, as the *Mathematicians* call it, *Homogeneum Comparationis*; and let the Relation of these be expressed among themselves by this Equation, *viz.*

$$nx + \frac{nn-1}{2\times 3} nx^3 + \frac{nn-1}{2\times 3} \times \frac{nn-9}{4\times 5} nx^5 + \frac{nn-1}{2\times 3} \times \frac{nn-9}{4\times 5} \times \frac{nn-25}{6\times 7} nx^7, \text{ \&c.} = a$$

It is manifest from the Nature of this Series, that if some unequal Number be taken for n (*viz.* an Integer, for it matters not whether it be Affirmative or Negative) then the Series will be limited, as above, whose Root is

$$1.\quad x = \tfrac{1}{2}\sqrt[n]{\sqrt{1+aa}+a} - \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}+a}},$$

or $$2.\quad x = \tfrac{1}{2}\sqrt[n]{\sqrt{1+aa}+a} - \tfrac{1}{2}\sqrt[n]{\sqrt{1+aa}-a}.$$

or $$3.\quad x = \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}} - \tfrac{1}{2}\sqrt[n]{\sqrt{1+aa}-a}.$$

or $$4.\quad x = \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}} - \frac{\frac{1}{2}}{\sqrt[n]{\sqrt{1+aa}-a}}.$$

Let

Let us find the Root of this Equation of the Fifth Power, *viz.*

$$5x + 20x^3 + 16x^5 = 4.$$

In which Case $n = 5$, and $a = 4$, the Root according to the first Form will be

$$x = \tfrac{1}{2}\sqrt[5]{\sqrt{17} + 4} - \frac{\frac{1}{2}}{\sqrt[5]{\sqrt{17} + 4}},$$ which may very expeditiously be solv'd in common Numbers; thus,

$\sqrt{17} + 4 = 8.1231$, whose Logarithm is 0.9097164, and the $\frac{1}{5}$ of 0.9097164 = 0.1819433, answering to $1.5203 = \sqrt[5]{\sqrt{17} + 4}$, and the Arithmetical Compliment of 0.1819433 is = 9.8180567, to which the Number 0.6577 is $= \frac{1}{\sqrt[5]{\sqrt{17} + 4}}$; therefore the Semidifference of those Numbers is $0.4313 = x$.

Here it is to be observed, that in the Place of the general Root x might very well be taken $= \frac{1}{2}\sqrt[n]{2a} - \frac{\frac{1}{2}}{\sqrt[n]{2a}}$, if at any Time the Number a, in Respect of Unity be greater, as if the Equation should be $5x + 20x^3 + 16x^5 = 682$, the Logarithm of $2a$ will be = 3.1348143, whose fifth Part is 0.6269628, and the Number answering thereto is 4.236. But the Arithmetical Compliment of 9.3730372 is 0.236, and the Semidifference of those Numbers is $2 = x$. But moreover,

If in the preceding Equation, the Signs alternately be affirmative and negative, as if the Series should happen after this Manner.

$nx +$

$nx + \frac{1-nn}{2\times3}nx^3 + \frac{1-nn}{2\times3}\times\frac{9-nn}{4\times5}nx^5 + \frac{1-nn}{2\times3}\times\frac{9-nn}{4\times5}\times\frac{25-nn}{6\times7}nx^7$, &c. $= a$, its Root will be, viz.

1. $x = \frac{1}{2}\sqrt[n]{a+\sqrt{aa-1}} + \frac{\frac{1}{2}}{\sqrt[n]{a+\sqrt{aa-1}}}$.

2. $x = \frac{1}{2}\sqrt[n]{a+\sqrt{aa-1}} + \frac{1}{2}\sqrt[n]{a-\sqrt{aa-1}}$.

3. $x = \frac{\frac{1}{2}}{\sqrt[n]{a-\sqrt{aa-1}}} + \sqrt[n]{a-\sqrt{aa-1}}$.

4. $x = \frac{\frac{1}{2}}{\sqrt[n]{a-\sqrt{aa-1}}} + \frac{\frac{1}{2}}{\sqrt[n]{a+\sqrt{aa-1}}}$.

And here it is to be noted, that if $\frac{n-1}{2}$ be unequal, the Sign of the Root found, must be contrary to it, let this Equation be proposed, viz. $5x-20x^3+16x^5=6$, whence $n=5$, and $a=6$, the Root will be $=\frac{1}{2}\sqrt[5]{6+\sqrt{35}} + \frac{\frac{1}{2}}{\sqrt[5]{6+\sqrt{35}}}$; or because $6+\sqrt{35}=11.916$, its Logarithm will be $=1.0761304$, and its $\frac{1}{5} = 0.2152561$, the Arithmetical Compliment 9.7847439; the Numbers of these Logarithms are 1.6415, and 0.6091 respectively, whose half Sum is $= 1.1253 = x$.

But if it should happen, that a is less than Unity, then the second Form of the Root, which is fitter for our Purpose, is to be chose above the rest; so if the Equation should be, viz. $5x-20x^3+16x^5=\frac{61}{64}$, x will be $=$

$$\frac{1}{2}\sqrt[5]{\frac{61}{64}+\sqrt{\frac{-375}{4096}}} + \frac{1}{2}\sqrt[5]{\frac{61}{64}-\sqrt{\frac{-375}{4096}}}$$

and

and indeed if the Fifth Root can be extracted by any Means, the true and possible Root will appear, although the very Expression itself seems to be impossible, and the $\frac{1}{5}$ Root of the Binomial $\frac{61}{64} + \sqrt{\frac{-375}{4096}}$ is $\frac{1}{4} + \frac{1}{4}\sqrt{-15}$, and likewise the $\frac{1}{5}$ Root of the Binomial $\frac{61}{64} - \sqrt{\frac{-375}{4096}}$ is $\frac{1}{4} - \frac{1}{4}\sqrt{-15}$, the half Sum of which Binomials is $= \frac{1}{4} = x$.

But if this Extraction cannot be had, or seems to be more difficult, it may very neatly be performed by a Table of Natural Sines, after the following Manner.

To the Rad. 1. let a be $= \frac{61}{64} = 0.95112$, the Sine of a certain Arc, which will be $= 72^{\circ}\ 23'$, whose fifth Part (because $n = 5$) is $14^{\circ}\ 28'$, the Sine of 0.24981 $= \frac{1}{4}$ nearly, and so we may proceed to Equations to a more superior Kind.

Q. E. I.

A METHOD *of approximating in extracting the Roots of Equations in Numbers.*

IN Philosophical Transactions, N°. 210, the late Dr. *Halley* has published a compendious and useful Method of extracting the Roots of *adfected* Equations of the common Form in Numbers. This Method proceeds by assuming the Root desired, nearly true to one or two Places in Decimals (which is done by Geometrical Construction, or some other convenient Way) and correcting the Assumption, by comparing the Difference between the true Root, and the assum'd, by Means of a new Equation, whose Root is the Difference, and which he shews how to form from the Equation proposed, by substituting the Value of the Root sought, partly in known, and partly in unknown Terms.

In doing this he makes use of a Table of Products (which he calls *Speculum Analyticum*) by which he computes the Co-efficients in the new Equation for finding the Difference mentioned. This Table, I observed, was form'd in the same Manner from the Equation proposed, as the Fluxions are, taking the Root sought for the only flowing Quantity, its Fluxion for Unity, and after every Operation dividing the Product successively by the Numbers 1. 2. 3. 4. &c.

Hence I soon found, that this Method might easily and naturally be made applicable, not only to Equations of the common Form (*viz.* such as consist of Terms, wherein the Powers of the Root sought are positive and integral without any Radical Sign) but also to all Expressions in general, wherein any Thing is proposed as given, which by any known Method might be computed; if *vice versa*, the Roots were considered as given: Such as are all Radical Expressions of Binomials, Trinomials, or of any other Nomial, which may be computed by the Root given, at least by Logarithms, whatever be the

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Index of the Power of that Nomial; as likewise Expressions of Logarithms, of Arches by the Sines or Tangents, of Areas of Curves by the *Abscissa's*, or any other Fluents, or Roots of Fluxional Equations, *&c.*

For the Sake of this great Generality, it may not be improper to shew how this Method is derived. Therefore z and x being two flowing Quantities (whose Relation to one another may be expressed by any Equation whatsoever, while z by flowing uniformly becomes $z+v$, x will become

$$x+\frac{\dot{x}}{1.\dot{z}}v+\frac{\ddot{x}}{1.2\dot{z}^2}v^2\mp\frac{\dddot{x}}{1.2.3\dot{z}^3}v^3+, \&c. \text{ or}$$

$$x+\frac{\dot{x}v}{1}+\frac{\ddot{x}v^2}{1.2}+\frac{\dddot{x}v^3}{1.2.3}+\&c. \text{ for } \dot{z} \text{ putting } 1.$$

Hence, if y be the Root of any Expression formed of y and known Quantities, and supposed equal to nothing, and z be a Part of y, and x be formed of z, and the known Quantities, in the same Manner, as the Expression made equal to nothing is formed of y, and let y be equal to $z+v$: the Difference of v will be found by extracting the Root of this Expression.

$$x+\frac{\dot{x}v}{1}+\frac{\ddot{x}v^2}{1.2}+\frac{\dddot{x}v^3}{1.2.3}+\&c.=0.$$ For in this Case z being become $z+v=y$, x which is now become

$$x+\dot{x}v+\frac{\ddot{x}v}{2}+\&c.$$ must become equal to nothing.

The Root v in the Equation $x+\frac{\dot{x}v}{1}+\frac{\ddot{x}v^2}{1.2}+\frac{\dddot{x}v^3}{1.2.3}+\&c.=0$, is to be found upon the Supposition

X 2 of

of its being very ſmall with Reſpect to z (as it muſt be if z be taken tolerably exact) by which Means the Terms $\frac{\dddot{x} v^3}{1.2.3} + \frac{\ddddot{x} v^4}{1.2.3.4}$ + &c. may be neglected upon Account of their Smallneſs with Reſpect to the other Terms, ſo as to leave the Equation $x + \frac{\dot{x} v}{1} + \frac{\ddot{x} v^2}{1.2} = 0$. for finding the firſt Approximation of v.

By extracting the Root of this Equation, we have

$$v = \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}}} - \frac{\dot{x}}{\ddot{x}}. \text{ That is,}$$

1	$\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}}} - \frac{\dot{x}}{\ddot{x}}$, if $x + \dot{x}v + \frac{\ddot{x} v^2}{2} = 0$.
2	$\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}}} - \frac{\dot{x}}{\ddot{x}}$, if $-x + \dot{x}v + \frac{\ddot{x} v^2}{2} = 0$.
3	$\surd\frac{\dot{x}}{\ddot{x}} - \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} - \frac{2x}{\ddot{x}}}$, if $x - \dot{x}v + \frac{\ddot{x} v^2}{2} = 0$.
4	$\surd\frac{\dot{x}}{\ddot{x}} - \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}}}$, if $-x - \dot{x}v + \frac{\ddot{x} v^2}{2} = 0$.

This Approximation gives v extract to twice as many Places as there are true Figures in z, and therefore trebles the Number of true Figures in the Expreſſion of y by $z+v$, which may be taken for a new Value of z, for computing a ſecond v, ſeeking other Values of x, $\dot{x}$, $\ddot{x}$, &c. Though, when z is tolerably exact (which it may be eſteem'd, when it contains two or three or more true Figures in the Value of y, according to the Number of Figures the Root is propoſed to be computed to,) the Calculation may be reſtored without ſo much Trouble, only

only by taking $\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} \pm \frac{2x}{\ddot{x}} - \frac{2\dddot{x}}{2.3\ddot{x}}v^3 - \frac{2\ddddot{x}}{1.2.3.4\ddot{x}}}$ v^4, &c. instead of $\sqrt{\frac{\dot{x}^2}{\ddot{x}^2} \pm \frac{2x}{\ddot{x}}}$, taking every Time for v its Value last computed.

From the same Equation $x + \dot{x}v + \frac{\ddot{x}v^2}{2} + \frac{\dddot{x}v^3}{1.2.3}$ $+$, &c. $= 0$, may be gathered also a Rational Form, *viz.* $v = \frac{-x}{\dot{x} - \frac{x\ddot{x}}{2\dot{x}}}$. For neglecting the Terms $\frac{\dddot{x}v^3}{1.2.3}$ &c. we have $v = \frac{-x}{\dot{x} + \frac{\ddot{x}}{2}v}$, which is nearly $= \frac{-x}{\dot{x}}$. Therefore in the Divisor instead of v writing $\frac{-x}{\dot{x}}$, we have more exactly $v = \frac{-x}{\dot{x} - \frac{x\ddot{x}}{2\dot{x}}}$. That is,

1	$\frac{-x}{\dot{x} - \frac{x\ddot{x}}{2\dot{x}}}$, when $x + \dot{x}v + \frac{\ddot{x}v^2}{2}$, &c. $= 0$.
2	$\frac{x}{\dot{x} + \frac{x\ddot{x}}{2\dot{x}}}$, when $-x + \dot{x}v + \frac{\ddot{x}v^2}{2}$, &c. $= 0$.
3	$\frac{x}{\dot{x} - \frac{x\ddot{x}}{2\dot{x}}}$, when $x - \dot{x}v + \frac{\ddot{x}v^2}{2}$, &c. $= 0$.

4. $-x$

$$4 \frac{-\dot{x}}{\dot{x} + \frac{\ddot{x}x}{2\dot{x}}}, \text{ when } -x - \dot{x}v + \frac{\ddot{x}v^2}{2} \&c. = 0.$$

This *Formula* will alſo triplicate the Number of true Figures in z. And the Calculation may be repeated after every Operation, taking for a Diviſor

$$\dot{x} \pm \frac{\ddot{x}}{2} v + \frac{\dddot{x}v^2}{1.2.3} + \frac{\ddddot{x}v^3}{1.2.3.4} +, \&c. \text{ inſtead of}$$

$$\dot{x} + \frac{\ddot{x}x}{2\dot{x}}.$$

Dr. *Halley* has fully explained the Manner of uſing both theſe *Formula's* in Equations of the common Form: Wherefore I ſhall be the ſhorter in explaining two or three Examples of another Sort.

EXAMPLE I.

Let it be propoſed to find the Root of this Equation, $\overline{y^2+1}|^{\sqrt{2}} + y - 16 = 0$. In this Caſe for y writing z, and for 0 writing x, we have $\overline{z^2+1}|^{\sqrt{2}} + z - 16 = x$; Whence by taking the Fluxions, we have $\dot{x} = 2\sqrt{2} \times z \times \overline{z^2+1}|^{\sqrt{2}-1} + 1$, and $\ddot{x} = 2\sqrt{2} \times \overline{8 - 4\sqrt{2z^2}} \times \overline{z^2+1}|^{\sqrt{2}-2}$. For finding the firſt Figures of the Root y for $\sqrt{2}$ take $\frac{3}{2}$, and we have the Equation $\overline{y^2+1}|^{\frac{3}{2}} + y - 16 = 0$, which being expanded, gives

$$y^6 + 3y^4 + 2y^2 + 32y - 255 = 0.$$

By this Equation I find, that for the firſt Suppoſition we may take $z = 2$. Therefore in order to find v, let us now make $\sqrt{2} = \frac{7}{5}$ (which is nearer than before) and we

have

have $x = \overline{z^2+1}|^{\frac{7}{5}} + z - 16 = \overline{2^2+1}|^{\frac{7}{5}} - 14 = 5^{\frac{7}{5}} - 14 = -4.48$; $\dot{x} = 10.66$; $\ddot{x} = 4.72$. Whence by the second Rational Form $v = \dfrac{4.48}{10.66 + \dfrac{4.72 \times 4.48}{2 \times 10.66}}$ $= 0.38$; which must be too big, because $\frac{7}{5} \sqsubset \sqrt{2}$, and therefore will require a larger Value of y to exhaust the Equation, than where $\sqrt{2}$ is exact. For the second Supposition therefore, let us take $z = 2.3$, and make $\sqrt{2} = 1.4142136$; and by the Help of the Logarithms, we shall have $\overline{z^2+1}|^{\sqrt{2}} = 13.47294$, whence $x = -0.22706$; $\dot{x} = 14.93429$, and $\ddot{x} = 5.18419$. Hence by the 2d Irrational *Formula* $v = \sqrt{\dfrac{14.93429^2}{5.18419^2} + \dfrac{0.45412}{5.18419}} - \dfrac{14.93429}{5.18419} = 0.01516$, which gives $y = z + v = 2.31516$, which is true to six Places. If you desire it more exact than to the Extent of the Tables of Logarithms, taking $z = 2.31516$ for the next Supposition, the Calculation must be repeated by computing of $\overline{zz+1}|^{\sqrt{2}}$ to a sufficient Number of Places, which must be done by the Binomial Series, or by making a Logarithm on Purpose, true to as many Places as are necessary.

EXAMPLE 2.

For another Example let it be required to find the Number whose Logarithm is 0.29: supposing we had no other Tables of Logarithms but Mr. *Sharp*'s of 200 Logarithms to a great many Places. This amounts to the resolving this Equation, $Ly = 0.29$, or $Ly - 0.29 = 0$; hence therefore we have $x = Ly - 0.29$, $\dot{x} = \frac{a}{y}$ (a being

ing the *Modulus* belonging to the Table we use, *viz.* 0.4342944819, &c.) $\ddot{x} = \frac{-a}{z^2}$, $\dot{\ddot{x}} = \frac{2a}{z^3}$, $\ddot{\ddot{x}} = \frac{-6a}{z^4}$, &c. In this Case, because $\ddot{x}$ has a Negative Sign, changing the Signs of all the Co-efficients, the Canon for v will be found in the fourth Case, which in the Irrational Form, gives $v = \frac{\dot{x}}{\ddot{x}} - \sqrt{\frac{\dot{x}^2}{\ddot{x}^2} + \frac{2x}{\ddot{x}} \frac{-2\dot{\ddot{x}}}{2.3\ddot{x}}v^3 - \frac{2\ddot{\ddot{x}}}{2.3.4\ddot{x}}v^4}$, &c. $= z - \sqrt{z^2 + \frac{2Lz - 0.58}{a} \times z^2 + \frac{2v^3}{3z} - \frac{2v^4}{4z^2} + \frac{2v^5}{5z^3}}$, &c. In this Case to avoid often dividing by z, it will be most convenient to compute $\frac{v}{z}$, which is got from this Equation $\frac{v}{z} = 1 - \sqrt{1 + \frac{2Lz - 0.58}{a} + \frac{2v^3}{3z^3} - \frac{2v^4}{4z^4} + \frac{2v^5}{5z^5}}$, &c. The nearest Logarithm, in the Tables proposed, to the proposed Logarithm 0.29, is 0.2900346114, its Number being 1.95. Therefore for the first Supposition taking $z = 1.95$, we have x $(= Lz - 0.29 = 0.2900346114 - 0.29) = 0.0000346114$, and $\frac{2Lz - 0.58}{a} = \frac{0.0000692228}{0.4342944819} = 0.00015939139$, and $1 + \frac{2Lz - 0.58}{a} = 1.00015939139$.

Whence for the first Approximation, we have $\frac{v}{z} = 1 - \sqrt{1.00015939139} = -0.00007969247$, and $v = -0.00015540032$, and $y = z + v = 1.94984459968$. which is true to eleven Places, and may easily be corrected by the Terms $\frac{2v^3}{3z}$, &c. which I leave to the *Reader's* Curiosity.

A General

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verging Ser *Power.*

om the Cube what Sign is different in your

s are wanting Theorem, as the following

stituting p, q, r, &c. are divided by	M S n.

A General Series *for expressing the Root of any* Quadratic Equation.

ANY Quadratic Equation being reduced to this Form, $xx - mnx + my = 0$, the Root x will be expressed by this Series of Terms, $x = \frac{y}{n} + A \times \frac{1}{\frac{mn^2}{y} - 2} + B \times \frac{1}{a^2 - 2}$ $+ C \times \frac{1}{b^2 - 2} + D \times \frac{1}{c^2 - 2}$, *&c.* which is thus to be understood.

1. The Capital Letters A, B, C, *&c.* stand for the whole Terms with their Signs, preceding those wherein they are found, as $B = A \times \frac{1}{\frac{mn^2}{y} - 2}$.

2. The little Letters a, b, c, &c. in the Divisors are equal to the whole Divisors of the Fraction in the Terms immediately preceding thus, $b = a^2 - 2$.

For an Example of this, let it be required to find $\sqrt{2}$, putting $\sqrt{2} = x + 1$, we have $x^2 + 2x - 1 = 0$, which being compared with the general *Formula*, gives $mn = -2$, and $my = -1$. Therefore for m taking -1, we have $n = 2$, and $y = 1$, which Values substituted in the Series, give $x = \frac{1}{2} - \frac{1}{2.6} - \frac{1}{2.6.34} - \frac{1}{2.6.34.1154}$ $- \frac{1}{2.6.34.1154.1331714}$, *&c.* The Fractions here wrote down giving the true Root to twenty-three Places. Q. E. I. & D.

CONSTRUCTION OF EQUATIONS.

IN order to this Method of Conftruction, I confider each Side of the Equation, as the Product of two Multipliers, the one of two Dimenfions, the other of one (each Term in a Cubic Equation being fuppofed of three Dimenfions).

EXAMPLE.

In this Equation $x^3+px^2=n$, I confider it as in this Form $xx\times\overline{x+p}=n=b^2\times c$ (b being taken for any Number at Pleafure, whofe Square is lefs than n divided by bb, gives c) or elfe in this Form, $\overline{xx+px}\times x=n=b^2\times c$; either of which Ways we may make ufe of, as feems lefs for Conftruction. And becaufe x is yet unknown, and muft be taken by Guefs, I put z, inftead of x, the Multiplier of two Dimenfions, and y for x in the other of one Dimenfion, and then the former will ftand thus, $zz\times\overline{y+p}=b^2\times c$, or (the other Way) $\overline{zz+pz}\times y=b^2\times c$. In both which Forms the given Quantity $b^2\times c=n$, is the fame as in the firft Equation, and confequently the Refult or Value of the other Terms is the fame alfo.

The Defign then of this our Method is, by taking a Number or Line by Guefs (fuppofe z) to reprefent x in one of the Multipliers of the given Equation, to find another Number or Line (y) which fhall reprefent x in the other Multiplier, and then if z and y be not equal, we we muſt bring them by Trials to Equality, which in moſt Caſes is eaſily done, obſerving their Difference, and the Nature of the Scheme or Figure.

Before I give Examples, I will premiſe this following *Lemma*, which ſhews the Ground and Demonſtration of this Way of Conſtruction.

Let ABD be a Semicircle (See *Fig.* 1.) on the Diameter AD, AB, and AZ, two Subtenſes drawn at Pleaſure from the End of the Diameter A, from B and Z are drawn the infinite Lines BC and ZS perpendicular to AD, BC interſecting it in *m*, and ZS in *n*; from A, draw the Line AS, interſecting BC in R, and ZS in S, I ſay, that $\overline{AB}^2 : \overline{AZ}^2 :: mR : NS$. *Fig.* 1

For by the Property of the Circle $DA \times mA = \overline{AB}^2$, and $DA \times nA = \overline{AZ}^2$, then $DA \times mA : DA \times nA :: mA : nA :: mR : nS$, that is, $\overline{AB}^2 : \overline{AZ}^2 :: Rm : nS$.

Multiply the Extremes and Mean together, and it will be $\overline{AB}^2 \times nS = \overline{AZ}^2 \times mR$; if therefore we ſuppoſe $AB = b$, $nS = c$; AZ = Square Root of the Multiplier of two Dimenſions (in a Cubic Equation reduced into the Form above directed) then will *m*R be equal to the other Multiplier of one Diviſion, ſo in the firſt Form above ($zz \times \overline{y+p} = b^2 \times c$) if $AZ = z$, then is $mR = y+p$; and in the ſecond Form $\overline{zz+pz} \times y = b^2 \times c$; if $AZ = \sqrt{zz+pz}$, then is $mR = y$; and if $y = z$, then $z = x$ in the given Equation.

EXAMPLE I.

Suppoſe I would conſtruct this Equation, $x^3 - 4x^2 = 72$, or $x^3 - px^2 = n$, I take 16, as a convenient, ſquare Number (which I call *bb*) and therewith I divide 72; the Quotient is $4\frac{1}{2}$, which I call *c* [$\frac{n}{bb} = c$, and $bbc = n = 72$] I deduce alſo the other Side of the Equation into two Multipliers (as above) and then it is $zz \times \overline{4-p} = 72 = 16 \times 4\frac{1}{2}$, which is the firſt Form for Conſtruction.

 I deſcribe

Fig. 1. I describe a Semicircle ABD (See *Fig.* 1.) of a convenient Bigness from my Scale of equal Parts which here, for this Figure, is of 24 in an Inch, 10 of which Parts make an Unit or 1.) and having drawn the Diameter AD, I take 4 (Units or large Divisions) off the Scale, and draw the Chord AB $= 4 = b$, from BI draw the Infinite BC, Perpendicular to AD, and interfecting it in m.

I take $4\frac{1}{2}$ ($= c$) off the Scale, and set that Distance with the Compasses from m to C, and through CI draw CS parallel to AD.

For the first Trial, I consider, that the Root x must be bigger than 4 or p (else the negative Term $-4xx$ would take more than xxx, and so the given Quantity would be negative) therefore taking 4 ($= p$) from the Scale; with the Centre A, and Radius A$p = 4$, I describe the little Arch Pp; and then (at a venture) draw the Chord Az ($= z$) intersecting the Arch Pp in p; so is A$p = p = 4$, and the Line $pz = z = p$, or $z - 4$. From z I draw ZS perpendicular to CS, and intersecting it in S, and then the Line AS intersecting MC in r. So is Mr $= y - 4$, or $y - p$ (the other Multiplier in the Equation) which being greater than the Line pz (to which it should be equal, it shews, that z was taken too little.

After the same Manner I try another z, which the View of the Scheme will now direct me to limit, till I find AZ, which answers the Demand; for making AZ $= z$, then is PZ ($= z - 4$) $=$ MR ($= y - 4$) consequently Z $= y = x$, z taken from the Scale is equal to 6, the Root sought.

The same Conclusion would follow, if I had inverted the Order of proceeding, and had begun with Mr, and thereby found Az (in a first Trial) for in this Case I must have taken a Line for y (by Guess) and made $mr =$ $y - p$, and then having drawn AS intersecting mC in r, and CS in S. Also SZ parallel to BC, touching the Semicircle in z; I draw Az, which will be equal to z, so is the Line $pz = z - p$, which ought to be equal to mr; but not being so, another Trial must be made.

EXAMPLE

EXAMPLE 2.

Let the Root of this Equation, *viz.* $xxx - 3xx + 2x = 24$, or $x^3 - p2x + qqx = n$. I take for bb, 9, by which dividing 24, it gives $C = 2\frac{2}{3}$, and then I put the Equation into this Form, $\overline{zz - pz + qq} \times y = bb \times c$.

In the Semicircle ABD (*Fig.* 2.) I draw the Chord *Fig.* $AB = b = 3$. BC perpendicular to AD, $mC = C = 2\frac{2}{3}$. CS parallel to AD. I find $\sqrt{zz - pz + qq}$, by *Fig.* 3. *Fig.* where $AP = p = 3$, and is bisected in C. $PQ = q = \sqrt{2} = 14$. Pd is perpendicular to PA, Az, AZ, &c. are Lines taken by Guess, for z. zd, ZD, &c. are Arches of Circles drawn with the Centre C, and Radius Cz, CZ, &c. so are Pd, pD, &c. $= \sqrt{zz - pz}$, and dQ DQ, &c. $= \sqrt{zz - pz + qq}$ (for continuing the Arch ZD (for Instance) to θ in the Diameter; ZA $(= P\theta) = z$. $PZ = z - p$. Therefore PD $(= \sqrt{P\theta \times PZ} = \sqrt{z \times \overline{z - p}} = \sqrt{zz - pz}$. And PQ being $= q$. $\overline{DQ}^2$ $(= \overline{DP}^2 + \overline{PQ}^2) = zz - pz + qq$; therefore $DQ = \sqrt{zz - pz + qq}$.

Having found $dQ = \sqrt{zz - pz + qq}$. I draw Az (See *Fig.* 2.) $-d$Q, and then (as in the former Example) *Fig.* 2. find $mr = y$, which being much less than z (or Az *Fig.* 3.) *Fig.* I find that I have erred in my Supposition of z. And considering that (See *Fig.* 3.) the bigger Az is, the bigger *Fig.* will dQ be also; and consequently mr the less: I try again with a lesser z, and at last find that making AZ (*Fig.* 3.) $= z$, DQ will be $\sqrt{zz - pz + qq}$, to which I *Fig.* make AZ (*Fig.* 2.) equal, and thereby find $mR = y = z$, *Fig.* which is therefore the Root, and the Scale shews the Number to be $6 = x$.

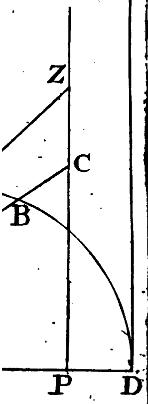
For another *Example* may be proposed, the doubling of the Cube, that is, having the Root or Side of a Cube given to find another Line, whose Cube shall be double the former Cube. In this Case, let AB (*Fig.* 1.) be *Fig.* the Side of the given Cube; $mC = 2AB$, $Az = z$, the sought Root taken by Guess, by which finding mr (as above)

if

if $mr = Az$, then is Az the Root of the double Cube sought; else another Trial must be made.

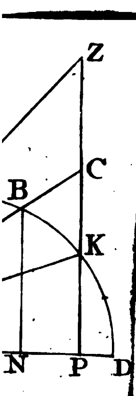
$\overline{AB}^2 : \overline{Az}^2 :: mr : nS = mc = 2AB$; therefore

$2AB \times \overline{AB}^2 = \overline{2AB}|^3 = \overline{Az}^2 \times mr = \overline{Az}|^3$, (when $Az = mr$).



By a much like Method may Biquadratic Equations be constructed also, if the lowest Term be wanting, as easily as a Cubic.

Suppose this Biquadratic Equation, $x^4 + px^3 \pm q^2x^2 = n$. I divide n by a less square Number; suppose b^2, the Quotient I call CC, $\frac{n}{bb} = CC$, then $n = b^2C^2$. (See *Fig.* 4.) I divide also the other Side of the Equation into two Multipliers, *viz.* $x^2 \times \overline{x^2 \pm px \pm qq} = b^2 \times C^2$, whence $x \times \sqrt{x^2 \pm px \pm qq} = b \times C$. Or, substituting y and z for x (as I do in the Cubes, while x is unknown) $q \times \sqrt{zz \pm pz \pm qq} = b \times C$. I take z by Guess, and therewith find $\sqrt{zz \pm pz \pm qq}$, as is done in the second Example of Cubic Equations). In a convenient Semicircle ABD (*Fig.* 4.) I apply the Chord $AB = b$, and producing it, make $AC = c$ [but if b be greater than c, I make $AB = c$, and $AC = b$] Through CI I draw ZCP, perpendicular to AD, and applying $AZ = \sqrt{zz \pm pz \pm qq}$, so is the intercepted Chord $AY = y$; and if $y = z$, then is either $= x$, the Root sought. Else Trial must be made with another z.



I might have taken the Biquadratic Root of n, and then $b - c$, to which the Diameter AD must have been equal: The Demonstration depends on this, that (*Fig.* 5.) $AY \times AZ = AB \times AC$, which I thus prove (having drawn AK to the Intersection of ZP with the Semicircle) $\overline{AB}^2 : \overline{AK}^2 :: (An : AP ::) AB : AC$. Therefore $AB \times AC = \overline{AK}^2$. By a like Reason.

$$\overline{AY}^2 : AK :: (AM : AP ::) AY : AZ.$$

Ergo $AY \times AZ = \overline{AK}^2 = AB \times AC$.

The same Method will hold for Biquadratics.

Generally.

Generally.

[illegible]

right Hand if negative. [illegible] the same is to be understood of AG and BR; which likewise must be drawn to contrary Parts, if the Values of r and s come out negative.

Lastly, on the Centre E, and with the Radius $EC = t$, let a Circle $CKxc$ be described, which shall cut the

Parabola

if $mr = Az$, then is Az the Root of the double Cube ſought; elſe another Trial muſt be made.

$\overline{AB}^2 : \overline{Az}^2 :: mr : nS = mc = 2AB$; therefore

$2AB \times \overline{AB}^2 = \overline{2AB}|^3 = \overline{Az}^2 \times mr = \overline{Az}|^3$, (when $Az = mr$).

By a much like Method may Biquadratic Equations

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: AP ::) AY : AZ.

= AB × AC.

ld for Biquadratics.

Generally.

Generally.

GIVEN any Equation, whether Cubic or Biquadratic, we must compare its Terms first.

As suppose this Equation.

$$x^4 = \frac{2p}{q} x^3 + \frac{4pr}{q1} x^2 + \frac{2p^2}{q} x + p^2.$$
$$- 4r \quad - 4r^2 \quad - \frac{2ps}{q} \quad - q^2.$$
$$+ 2s \quad + 4rs \quad - s^2.$$
$$- 1 \quad - 2q \quad + t^2.$$

By which Means p, q, r, s, t, will easily be found; in the mean time, let some one be taken at Pleasure.

Then in any given Parabola AVB (See *Fig.* 6.) whose *Fig.* 6. principal Vertex is V, Axis VS, and VT perpendicular to the Axis; Let VS be taken $= p$, within the Parabola, and let ST$= q$ be inscribed in the Angle SVT, which being produced, may cut the Parabola in two Points N and O. Bisect ON in M, and thro' M draw MA parallel to the Axis, meeting the Parabola in A, then draw AL parallel to ON, that AL may be the *Latus Rectum* of the Parabola to the Diameter AM, and let the same be $=$ Unity. In AL (if it is necessary to produce on each Side) let AG be taken $= r$, and from the Point G, let GR be drawn parallel to the Axis, which may cut the Parabola in B, from which let BR $= s$. From R the Point just found, let RE be drawn parallel and equal to VT, which must be drawn to the left Hand in Respect of R, if q be affirmative, but to the right Hand if negative. And the same is to be understood of AG and BR; which likewise must be drawn to contrary Parts, if the Values of r and s come out negative.

Lastly, on the Centre E, and with the Radius EC$=t$, let a Circle CKxc be described, which shall cut the

Parabola in as many Points, as there are real Roots in the given Equation.

For from the Points C, K, &c. draw CP, KΠ, &c. parallel to ST, terminated at the right Line GR, (if it seems necessary to be produced) and any of these will be x, or the Root sought of the given Equation.

Those drawn on the right Hand will be affirmative Roots, but those on the left will be negative Roots. The Point of Contact, if there shall be one, is here taken for the two Points of Intersection nearest to one another.

Betwixt Cubic and Biquadratic Equations so constructed this Difference only arises, that in the former Equations, by Reason of the last Term being wanting in the preceding Equation, is always $p^2 - q^2 - s^2 + t^2 = 0$, or $t = \overline{s^2 + q^2 - p^2}\big|^{\frac{1}{2}}$. Therefore from the Centre E, and with the Radius EB $(= \sqrt{\overline{BK}^2 + (\overline{ER}^2)\ \overline{ST}^2 - \overline{VS}^2})$ having described the Circle CKxc, one of the Roots in the former Equation vanishes to nothing.

And these are demonstrated after the following Manner. Having constructed the continued Points, and CP produced, if there is Occasion for it, till it cut AM in H, CH will be the ordinate of the Parabola to the Diameter AH, and therefore $\overline{CH}^2 = AL \times AH = AH$, because $AL = 1$. But $CH = CP + AG$, and $AH = GB + BP$, and therefore $\overline{CP}^2 + 2AG \times CP + \overline{AG}^2 = GB + BP$. Now from the Point C, let fall the Perpendicular CD to BP, which meets EI in the Point I drawn parallel to BP. Therefore by similar Triangles CDP and TVS, it will be

$$ST : VS :: CP : \frac{VS \times CP}{ST} = DP.$$

$$\text{And} \quad ST : VT :: CP : \frac{VT \times CP}{ST} = CD.$$

And therefore $\overline{CP}^2 + 2AG \times CP = BP = DP + BD = \frac{VS \times CP}{ST} + BR - IE$, or $\overline{CP}^2 + 2AG \times CP - \frac{VS}{ST} CP - BR = -IE$. But $\overline{IE}^2 = \overline{CE}^2 - \overline{CI}^2 = \overline{CE}^2 - \overline{CD}^2 - \overline{VT}^2$.

$$-\overline{VT}^2 - 2CD \times VT = \overline{CE}^2 - \frac{\overline{VT}^2 \times \overline{CP}^2}{\overline{ST}^2} - \overline{VT}^2 -$$

$$\frac{2\overline{VT}^2 \times CP}{ST} = (\text{because } \overline{VT}^2 = \overline{ST}^2 - \overline{SV}^2)\ \overline{CE}^2 - \overline{CP}^2$$

$$+ \frac{\overline{SV}^2}{\overline{ST}^2}\overline{CP}^2 - \overline{ST}^2 + \overline{SV}^2 - 2ST \times CP + \frac{2\overline{SV}^2}{ST} CP,$$

therefore which will be equal to the Square of the Side $\overline{CP}^2 + 2AG \times CP - \frac{VS}{ST} CP - BR$. And this Equation is reduced to the Terms p, q, r, s, t, as the very Equation was propoſed. Q. E. D.

Hence it appears, that any Biquadratic Equation may be conſtructed innumerable Ways by the Parabola, for an indefinite Value of that Quantity, which we ſaid was aſſumed at Pleaſure. But the Caſe is the moſt ſimple, by making $VS = p = 0$, and then the Conſtruction is changed, if you regard the Thing into this common one, in which the right Lines CP, &c. are Repreſentatives of the Roots Perpendiculars to the Axis. And the Equation becomes

$$\begin{array}{llll} x^4 = -4rx^3 & -4r^2x^2 & +4rsx & -q^2 \\ & +2s & -2q & -s^2 \\ & -1 & & +t^2 \end{array}$$

which is eaſily conſtructed as above.

Some Properties of CONIC SECTIONS.

Fig. 7. LET DE be the Tranſverſe Axis of the Ellipſis (See *Fig.* 7.) AO the other Axis, and C the Centre of the Section: Let P be any Point in its Circumference; PQ the Tangent of the Curve at P, meeting the tranſverſe Axis at Q, the Points of the Foci SF; CP, CK, of the Semi-conjugate Diameter. PH the *Semilatus Rectum* to the Diameter PC; PG, the Normal to the Tangent, which meets HG in the Point G, Perpendicular to PCH, that PG, the Radius of the Curve of the Ellipſis, may be in the Point P; and let ST, CR, FV, be Perpendiculars let fall on the Tangent PQ; let SO be join'd, and the Normal PL let fall on the Axis, this being put, I ſay, that

I.

The Rectangle under the Diſtances from each Focus of the Ellipſis, or SP×SF *is equal to the Square of the Semidiameter* CK.

DEMONSTRATION.

$\overline{PS}^2 = \overline{PC}^2 + \overline{CS}^2 - 2CS \times CL$. By 13 Euc. 2.

$\overline{PF}^2 = \overline{PC}^2 + \overline{CS}^2 + 2CS \times CL$. By 12 Euc. 2.

Whence $\overline{PS}^2 + \overline{PF}^2 = 2\overline{PC}^2 + 2\overline{CS}^2$.

Now $PS + PF = DE = 2CD$, and conſequently

$\overline{PS}^2 + \overline{PF}^2 + 2PS \times PF = 4\overline{CD}^2$.

Wherefore by tranſpoſing $2PS \times PF = 4\overline{CD}^2 - 2\overline{PC}^2 - 2\overline{CS}^2$

And by halving it, $PS \times PF = 2\overline{CD}^2 - \overline{PC}^2 - \overline{CS}^2$.

And $\overline{CS}^2 = \overline{CD}^2 - \overline{CO}^2$. and conſequently

$PS \times PF = \overline{CD}^2 + \overline{CO}^2 - \overline{PC}^2$.

But $\overline{CD}^2 + \overline{CO}^2 = \overline{PC}^2 + \overline{CK}^2$. By 12. 7 *Apoll. Con.*

Conſequently, $PS \times PF = \overline{CK}^2$. Q. E. D.

II. *The*

II.

The Distance from the Focus SP *is to the Perpendicular let fall on the Tangent, as the Semi-conjugate Diameter* CK *to the lesser Semi-axis* CO.

Demonstration.

By Reason of the Similar Triangles SPT, FPV, it will be, PS : PF :: ST : FV, and by comparing them, PS+PF will be to ST+FV, and the half of CD to CR, as PS to ST; whence CD × CK will be to CR × CK, as PS to ST. But CR × CK is equal to the Rectangle under the Semi-axes CD into CO, by 31. VII. *Conic.* Consequently PS is to ST, as CD into CK is to CD × CO, or, as CK to CO. And by a like Argument PF will be demonstrated to be to FV in the same Ratio.

Q. E. D.

III.

Likewise in the same Ratio is the Semi-axis Transversus CD *to the Normal, from the Centre* C, *let fall to the Tangent, or to* CK.

For seeing the Rectangle CR × CK is equal to the Rectangle CD × CO, as already aforesaid (ἀνάλογον) the Analogy will be CD to CR, as CK to CO.

Q. E. D.

IV.

Every Semidiameter PC *is to the Distance of the Point* P *from the Focus* S, or to SP, *as the Distance from the other Focus* FP, *to half the* Latus Rectum, *reaching to the* Vertex P, *or as to* PH.

This is manifest from Proposition 1. namely, when the Square of CK is equal to the Rectangle under SP × PF.

V.

The Rectangle of the Semi-axes CD × CO *is to the Square of the Semi-conjugate Diameter* CK, *as* CK *to the Radius of the Curve in the Point* P, *or to* PG.

 For

For the Triangles PCR, PGH, are similar amongst themselves; whence CR is to PC, as the *Semilatus Rectum* PH to PG. By III. premised Prop. $\frac{CD \times CO}{CK} = CR$ is to PC, as $\frac{\overline{CK}^2}{PC} = PH$ to $\frac{CK^3}{CD \times CO} = PG$, wherefore the Analogy $CD \times CO : \overline{CK}^2 :: CK : PG$. Q. E. D.

Hence this General Theorem I.

The Centripetal Force tending to the same Point S in all Curves, is always proportional to the Quantity $\frac{SP}{PG \times ST^3}$.

Now by writing CK^3 for PG, by Proposition V. and $\frac{SP}{CK}$, according to Proposition II. for ST (*viz.* because CD, CO is given) the Centripetal Force tending to the Focus S of the Ellipsis, will always be, as $\frac{SP \times CK^3}{\overline{CK}^3 \times \overline{SP}^3}$ that is, as $\frac{SP}{SP^3}$, or $\frac{1}{SP^2}$, *viz.* reciprocally, as the Square of SP. Whence it appears, that if the Section described by the Motion of the Body be an Ellipsis, the Centripetal Force will be as the Square of the Distance from the Centre of Force reciprocally. Hence arise these useful Corollaries.

COROLLARY I.

The Velocity of a Body revolving in the Ellipsis to any Point P, *is to the Velocity of a Body revolving in the Circle to the same Distance* SP *from the Centre of Force, in* Subduple *Ratio of the Distance from the other Focus* PF, *to the* Semiaxis Transversus *of the Section, or as a mean Proportional between* PF, *and*, CD, *to* CD.

For the Velocity of a Body revolving in the Ellipsis to the Distance SP is to the Velocity of a Body revolving in a Circle or Ellipse, to the Distance of the Semiaxis CD, or SO, as CO to ST, that is by Proposition II. as $\sqrt{PF}$ to $\sqrt{SP}$: and the Velocity of a Body revolving in a Circle a Circle to the Distance CD, is to the Velocity of a Body revolving in a Circle to the Distance SP, as $\sqrt{SP}$ to $\sqrt{CD}$. Therefore, *ex æquo*, the Velocity to the Distance SP, is to the Velocity of a Body revolving in a Circle to the same Distance, as $\sqrt{PF}$ to $\sqrt{CD}$.

Corollary 2.

From the Velocity in the Ellipsis, the Position of the Tangent, and the Centre of Force or Focus being given, it is easy to determine the other Focus.

For let the given Velocity be R; and that Velocity by which a Circle is to be described, to the given Distance SP from the Centre, let it be Q, and by the preceding Corollary, R is to Q, as $\sqrt{PF}$ to $\sqrt{CD}$; therefore QQ is to RR, as CD to PF, and 2QQ—RR will be to RR, as SP to PF. But SP is given, therefore PF is given in Magnitude; likewise it is given by Position, because the Angle VPF is equal to the Angle SPT. Consequently the Point F, the other Focus, is given, which being found, it is easy to describe the Section.

But if $\frac{1}{2}$ RR (See *Fig.* 8.) be greater than the Square of Q, 2QQ—RR is made a negative Quantity, and the Trajectory to be described in the *Place* of the Ellipsis passes thro' the Hyperbola, and RR—2QQ will be to RR, as SP to PF, the Distance of the other *Focus*, to be drawn on the other Side of the Tangent, as the Focus F is: But all the Properties, which we have demonstrated in the Ellipsis, *mutatis mutandis*, likewise agrees with the *Hyperbola*. *Fig.* 8.

But if it happen, that QQ is equal to half the Square of R, the Quantity 2QQ—RR vanishing = 0, the fourth Proportional PF is infinite; wherefore the Trajectory to be described is the Parabola, *viz.* in the other Focus passing *ad infinitum*. But the Axis of the Trajectory is given by Position; for it is parallel to PF, *viz.* the Angle FPV is equal to the given Angle SPT.

Corollary 3.

The Velocity of a Body revolving in a given Conic Section to the Distance SP, *is to its Velocity to the other*

other Distance SX, *as a mean Proportional between* FP, *and* SX, *to a mean Proportional between* SP *and* FX.

For the Velocity in P is as $\sqrt{\frac{FP}{SP}}$ (*per* Proposit. II.) and by the same, the Velocity X is as $\sqrt{\frac{FX}{SX}}$, whence the Proposition is manifest.

COROLLARY 4.

Likewise the Ratio of the Velocities of two Bodies revolving in the same System, but in given different Conic Sections, having given the Distances of both Orbits from the common Focus of the Orbits, will immediately be obtained by the Help of the first Corollary.

For seeing the Velocity of a Body in P is to the Velocity in the Circle, to the same Distance SP, as $\sqrt{PF}$ to $\sqrt{CD}$, and in any other supposed Conic Section, whose Semiaxis *cd*, and Foci F. *f*, to the Distance SP, those Velocities are as $\sqrt{pf}$ to $\sqrt{cd}$, and the Velocity of a Body revolving in a Circle to the Distance SP, is to the Velocity in the Circle to the Distance S*p*, as $\sqrt{Sp}$ to $\sqrt{SP}$, having *compared* the Ratios the Velocity in P will be to the Velocity in *p*, as $\sqrt{PF \times cd \times Sp}$ to $\sqrt{pf \times CD \times SP}$. But if the other Section be a *Parabola*, *cd*, *pf*, will be infinite, but in Ratio, as 1 to 2; consequently the Ratio of the Velocities will be, as $\sqrt{PF \times Sp}$ to $\sqrt{2CD \times SP}$.

COROLLARY 5.

But if in the Hyperbola, the Point p be drawn ad infinitum; *it is manifest from the preceding Corollaries, that the least Velocity, as the least, in which, when a Body ascends* ad æternum, *is equal to that, by which it should describe a Circle at the Distance* CD, *equal to the Semitransverse Axis.*

COROLLARY 6.

Having the Distance given from the Focus, the Position of the Tangent is also given, or the Angle SPT *contained under the Distance* S*p*, *and the Tangent* PT.

For

For it is (*per* Propofition II.) PS : ST :: CK : CO, or, as $\overline{SP \times PF}|^{\frac{1}{2}}$ to CO, and fo is Radius to the Sine of the Angle SPT. But in Ellipfes near to Circles it is better to make the Angle PST, its Compliment to the Square, and its Sine is to the Radius, as $\sqrt{SP \times PF - \overline{CO}^2}$ to $\sqrt{SP \times PF}$.

COROLLARY 7.

From hence flow the Velocities, by which the Diftances SP *increafe or decreafe.*

For feeing from the preceding Corollary $\sqrt{SP \times PF}$ is to $\sqrt{SP \times PF - \overline{CO}^2}$, as Radius to the Sine of the Angle PST, and in the fame Ratio is the Velocity of the Body in P to the Velocity of the fame Moment SP, and that Velocity may be in P (by Propofition II.) as $\sqrt{\frac{PF}{SP}}$; and cutting off the Overplus $\sqrt{\frac{SP \times PF - \overline{CO}^2}{SP}}$ will be to the Velocity, in which the Diftance SP increafeth, always proportional.

Hence this Second General Theorem.

In every Curvilinear Trajectory, the Angular Velocities about the Centre of Force are reciprocally proportional to the Squares of their Diftances from the Centre.

For by Reafon of equal Areas of the leaft Sections, the Subtenfe Arcs or Bafes to the leaft Angles are reciprocally as their Radii, and the leaft Angles, whereby equal Bafes are fubtended, are alfo reciprocally as their Radii. Confequently the Angles of the leaft Sections, equal in Area, are amongft themfelves reciprocally in duplicate Ratio, of the Radii, or, as the Square of their Diftances.

COROLLARY 8.

Hence the Angular Velocities of Bodies revolving in diverfe given Ellipfes are compared *among themfelves.*

For

For the Angular Velocities with which equal Circles may be described at the Distances to the Semiax. Transverses are reciprocally in *Sesquialter* Ratio of the Axes, or, as $\frac{1}{CD\sqrt{CD}}$. But the Bodies revolving has the middle Angular Velocities, when the Squares of their Distances are equal to the Rectangles under the Semiaxes of the Ellipses. Consequently (*per* Theorem II.) $\overline{SP}^2$ will be to $CD \times CO$, as $\frac{1}{CD\sqrt{CD}}$ to $\frac{CO}{\overline{SP}^2 \times \sqrt{CD}}$: Which Quantity is as the Velocity of the Angle at the Centre S, described by the Motion of the right Line SP, as the least Time given.

Corollary 9.

The Angular Velocity by which the Tangent PT *is circumgyrated, or the Perpendicular* ST *on the Tangent is to the Angular Velocity of the right Line* SP, *as the Semiaxis Transversus* CD *to the Distance from the other Focus* PF.

Demonstration.

Fig. 9. Let the Points be P,*p* (See *Fig.* 9.) near between themselves, and having drawn SP, S*p*, let PT, *pt*, be two Tangents, to which left fall the Perpendiculars ST, S*t*, and let the Radii of the Curve PG be drawn parallel to them, meeting *p*G in G; and on the Centre S with the Radius SP, let the small Arch PE meeting SP in E be described. It is manifest, that the Angle PG*p* is equal to the Angle TS*t*, or to the Angular Velocity of the Normal ST; and the Angle PS*p* is the Angular Velocity of the right Line S*p*; consequently the Angle PG*p* is to the Angle PS*p*, as the Angular Velocity of ST is to the Angular Velocity of the right Line SP, that is, as $\frac{Pp}{PG}$ to $\frac{PE}{PS}$. But P*p*:PE :: SP : ST :: CK : CO (*per* Proposition II.) Consequently therefore the Velocities

are

are as $\frac{C}{P}$

Propositio

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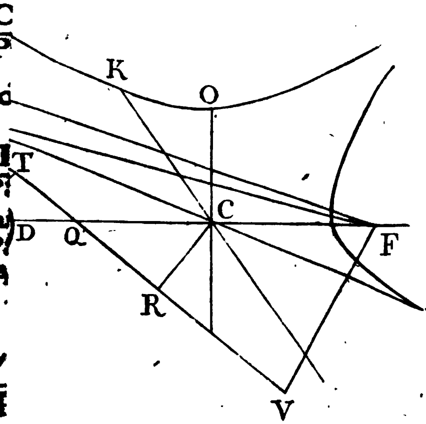
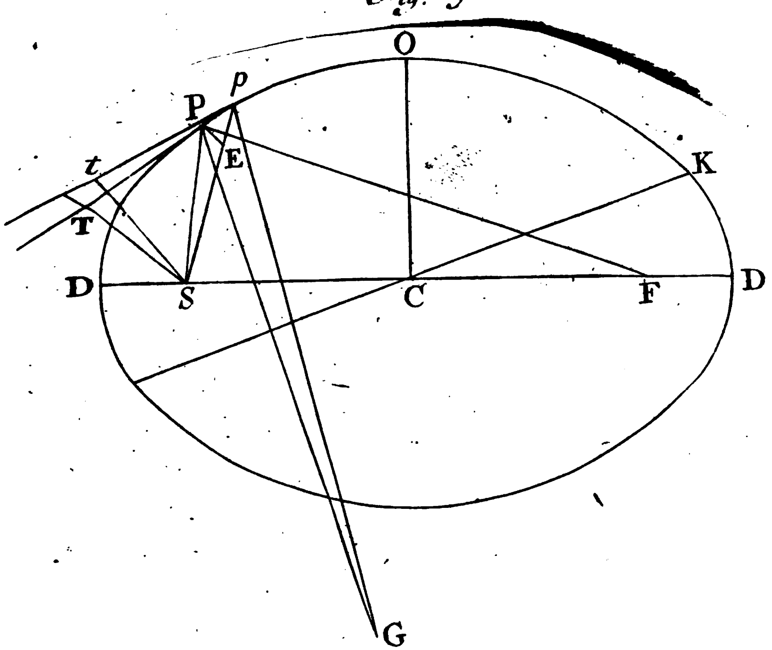


Fig: 9



Of Arcs, Sines, Cosines, &c. *converging.*

OUR great and incomparable Philosopher, Sir *Isaac Newton*, was the first that laid down a Series converging, *in infinitum*, from which having the Arcs given, their Sines may be found.

Thus, call the Arc A, and the Radius be an Unit, the Sine thereof will be found to be

$$A - \frac{A^3}{1.2.3} + \frac{A^5}{1.2.3.4.5} - \frac{A^7}{1.2.3.4.5.6.7} + \frac{A^9}{1.2.3.4.5.6.7.8.9}, \&c.$$

And the Cosine.

$$1 - \frac{A^2}{1.2.3} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8}, \&c.$$

These Series in the Beginning of the Quadrant, when the Arc A is but small, soon converge. For in the Series for the Sine, if A does not exceed 10 Minutes, the first Terms thereof, *viz.* $A - \frac{1}{6}A^3$ gives the Sine to 15 Places of Figures. If the Arc A be not greater than one Degree, the three first Terms will exhibit the Sine to 15 Places of Figures, and so the said Series are very useful for finding the first and last Sines of the Quadrant; but the greater the Arc A is, the more are the Terms of the Series required to have the Sine in Numbers true to a given Place of Figures; and then, when the Arc is nearly equal to the Radius, the Series converges very slow; and therefore to remedy this, I have devised other Series, similar to the *Newtonian* One, wherein, I suppose, the Arc, whose Sine is sought, is the Sum and Difference of two Arcs, *viz.* A+z, or A−z: And let the Sine of the Arc A be called *a*, and the Cosine *b*, then the Sine of the Arc A+z will be expressed thus:

Sine,

SINE.

$$1.\ a + \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5},\ \&c.$$

And COSINE.

$$2.\ b - \frac{az}{1} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} - \frac{az^5}{1.2.3.4.5} - \frac{bz^6}{1.2.3.4.5.6},\ \&c.$$

In like Manner the Sine of the Arc A—z is

$$3.\ a - \frac{bz}{1} - \frac{az^2}{1.2} + \frac{az^3}{1.2.3} + \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6},\ \&c.$$

COSINE is

$$4.\ b + \frac{az}{1} - \frac{bz^2}{1.2} - \frac{az^3}{1.2.3.} + \frac{bz^4}{1.2.3.4} + \frac{az^5}{1.2.3.4.5},\ \&c.$$

The Arc A is an Arithmetical Mean between the Arc A—z, and A+z, and the Difference of the Sines are

$$5.\ \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6},\ \&c.$$

$$6.\ \frac{bz}{1} + \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} + \frac{az^6}{1.2.3.4.5.6},\ \&c.$$

Whence the Difference of the Differences, or second Difference.

 7. Produce

7. Produce $\frac{2az^2}{1.2} - \frac{2az^4}{1.2.3.4} + \frac{2az^6}{1.2.3.4.5.6}$, &c.

Or $2a \times \frac{z^2}{1.2} - \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6}$, &c.

which Series is equal to double the Sine of the mean Arc drawn into the versed Sine of the Arc z, and converges very soon: So that if z be the first Minute of the Quadrant, the first Term of the Series, gives the second Difference to 15 Places of Figures, and the second Term to 25.

From hence, if the Sines of the Arcs distant one Minute from each other be given. The Sines of all the Arcs, that are in the same Progression, may be found by an exceeding easy Operation.

In the first and second Series, if $A=0$, then shall $a=c$, and b its Cosine will become Radius, or 1. And hence, if the Terms wherein a is, are taken away, and 1 be put instead of b, the Series will become *Newtonian*. In the third and fourth Series, if A be 90 Degrees, we shall have $b=0$, and $a=1$, whence again taking away all the Terms wherein b is, and putting 1, instead of a, we shall have the *Newtonian* Series arise.

But perhaps the young Reader will say, how, and by what Method, get we the above Series here, perhaps may be a Matter of Difficulty to the young Reader, but to those that are acquainted with the sublime Parts are readily conversant therein. And therefore, for the Sake of my young Reader, I shall give him here a brief Summary, how, and by what Methods, according to the Rules of Trigonometry, the Series for Sines, Cosines, Tangents, &c. are found from an Arc first given.

If Radius $= 1$, and x be its right Sine, then will $\sqrt{1-xx}$ be its Cosine.

DEMONSTRATION.

Fig. P. Draw the Quadrant ABC (See *Fig.* P) making A the Centre, AC = AS = 1, the Radius. CS $= a$ any

Arc,

Arc, therefore Sm, its right Sine, $= x$, then will Am, the Cosine be $= \sqrt{1 - xx}$, by 47 E. 1. for $\overline{AS}^2 - \overline{Sm}^2 = \overline{Am}^2$.

Again, let AR be another Radius of the Circle, infinitely near to AS, then will R$n = \dot{x}$ be the Fluxion of the Sine x, and the infinitely small Arc RS $= \dot{a}$, the Fluxion of the Arc CS; and because the Triangles ASm, and RnS, are similar, it will be as $\sqrt{1 - x^2} : 1 :: \dot{x} : \dot{a}$ ∴

$$\dot{a} = \frac{\dot{x}}{\sqrt{1 - xx}}.$$ The Fluxion of the Arc CS.

Methinks it will not be amiss to explain to those that are yet but Beginners, how to extract the Square Root of $\sqrt{1 - xx}$, as this

OPERATION.

$$1 - xx\,(1 - \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 - \tfrac{1}{16}x^6 - \tfrac{5}{128}x^8, \text{ \&c.}$$

$$1$$

$$2 - \frac{x^2}{2}\Big) - xx$$

$$-xx + \tfrac{1}{4}x^4$$

$$2 - x^2 - \tfrac{1}{8}x^4) - \tfrac{1}{4}x^4$$

$$-\tfrac{1}{4}x^4 + \tfrac{1}{8}x^6 + \tfrac{1}{64}x^8$$

$$2 - x^2 - \tfrac{1}{4}x^4, \text{ \&c.}) - \tfrac{1}{8}x^6 - \tfrac{1}{64}x^8$$

$$-\tfrac{1}{8}x^6 - \tfrac{1}{16}x^8$$

$$-\tfrac{5}{64}x^8$$

Here you see the Square Root of $1 - xx$ is $= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8$, &c. But the Arc is $\dot{a} = \dfrac{\dot{x}}{\sqrt{1 - xx}}$,

Now, if the Fluxion of the Arc SC be divided by this Root, the Quotient is the Fluxion of the Arc SC.

OPERATION.

OPERATION.

$1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6$, &c.) $\dot{x}$ ($\dot{x}+\frac{1}{2}x^2\dot{x}+\frac{3}{8}x^4\dot{x}+\frac{5}{16}x^6\dot{x}$, &c.

$\dot{x}-\frac{1}{2}x^2\dot{x}-\frac{1}{8}x^4\dot{x}-\frac{1}{16}x^6\dot{x}$, &c.

$+\frac{1}{2}x^2\dot{x}+\frac{1}{8}x^4\dot{x}+\frac{1}{16}x^6\dot{x}$, &c.

$+\frac{1}{2}x^2\dot{x}-\frac{1}{4}x^4\dot{x}+\frac{1}{8}x^6\dot{x}$, &c.

$+\frac{3}{8}x^4\dot{x}+\frac{1}{8}x^6\dot{x}$, &c.

$+\frac{3}{8}x^4\dot{x}-\frac{3}{16}x^6\dot{x}$, &c.

$+\frac{5}{16}x^6\dot{x}$, &c.

Consequently this last Quotient $\dot{x}+\frac{1}{2}x^2\dot{x}+\frac{3}{8}x^4\dot{x}+\frac{5}{16}x^6\dot{x}$, &c. is equal to $\dot{a}$, the Fluxion of the Arc, and its flowing Quantity is equal to $x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7$, &c.

Therefore $a=x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7$, &c.

Or $a=x+x\,\frac{1\times1}{2.3}x^3+\frac{1.1.3.3}{2.3.4.5}x^5+\frac{1.1.3.3.5.5}{2.3.4.5.6.7}x^7$, &c.

Or $a=x+\frac{1.1}{2.3}x^3+\frac{1.3}{2.4.5}x^5+\frac{1.3.5}{2.4.6.7}x^7$, &c.

Hence the Length of any Sine or Ordinate in the Circle being given, the corresponding Arc may be readily found.

Now if the Root of the Infinite Equation $x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7$, &c. be extracted, we shall have have the Value of x expressed in the Terms of a, and consequently a direct Method for finding the Sine of any Arc, its Length being first given.

If $a=x+\frac{1}{6}x^3+\frac{3}{40}x^5+\frac{5}{112}x^7$, &c.

Then $x=a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{5040}a^7$, &c.

Or $x=a-\frac{1}{2.3}x^3+\frac{1}{2.3.4.5}x^5-\frac{1}{2.3.4.5.6.7}x^7$, &c.

For

$$\left.\begin{array}{l} \text{For put } x = Aa + Ba^3 + Ca^5, \text{ \&c.} \\ \text{Then will } \frac{1}{6}x^3 = \frac{1}{6}A^3a^3 + \frac{1}{2}A^2Ba^5, \text{ \&c.} \\ \text{And } \frac{3}{40}x^5 = \qquad + \frac{3}{40}A^5a^5, \text{ \&c.} \end{array}\right\} = a.$$

And consequently $Aa = a$, and $A = 1$; also $B + \frac{1}{6}A^3 = 0$, and $B = -\frac{1}{6}A^3 = -\frac{1}{6}$.

Also $C + \frac{1}{2}A^2B + \frac{3}{40}A^5 = 0$, and $C = -\frac{1}{2}A^2B - \frac{3}{40}A^5 = -\frac{1}{2} \times -\frac{1}{6} - \frac{3}{40} = +\frac{1}{12} - \frac{3}{40} = +\frac{1}{120}$.

Wherefore $A = 1$, $B = -\frac{1}{6}$, $C = +\frac{1}{120}$, &c. and consequently $x = a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. W. W. D.

COROLLARY.

Wherefore, if a be put for the Length of any Arc, the right Sine will be $a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c.

Again.

The Sine of any Arc being given, its Cosine may be had thus by Investigation.

For according to 47 E. 1. if from the Square of the Radius $= 1$. be taken the Square of the right Sine $= a - \frac{1}{6}a^3 + \frac{1}{120}a^5$, &c. the Square Root of the Remainder will be the Cosine $= 1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \frac{1}{720}a^6$, &c. thus,

$$\begin{array}{rl} x = & a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\ x = & a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c.} \\ \hline & a^2 - \frac{1}{6}a^4 + \frac{1}{120}a^6, \text{ \&c.} \\ & \quad - \frac{1}{6}a^4 + \frac{1}{36}a^6, \text{ \&c.} \\ & \qquad\quad + \frac{1}{120}a^6, \text{ \&c.} \\ \hline xx = & a^2 - \frac{1}{3}a^4 + \frac{2}{43}a^6, \text{ \&c.} \end{array}$$

which taken from the Square of the Radius 1. leaves $1 - aa + \frac{1}{3}a^4 - \frac{2}{45}a^6$, &c. the Square Root of which will be the Cosine, as appears from this

OPERATION.

Operation.

$$1)1-aa+\tfrac{1}{3}a^4-\tfrac{2}{45}a^6,\&c.(1-\tfrac{1}{2}a^2+\tfrac{1}{24}a^4-\tfrac{1}{170}a^6,\&c.$$

$$1$$

$$1-aa)-aa+\tfrac{1}{3}a^4$$

$$-aa+\tfrac{1}{4}a^4$$

$$2-aa,\&c.)+\tfrac{1}{12}a^4-\tfrac{2}{45}a^6,\ \&c.$$

$$+\tfrac{1}{12}a^4-\tfrac{1}{24}a^6,\ \&c.$$

$$2-aa,\ \&c.)-\tfrac{1}{360}a^6,\ \&c.$$

$$-\tfrac{1}{360}a^6.\ \&c.$$

Now if for the Cosine we put C, then

$$C=1-\tfrac{1}{2}a^2+\tfrac{1}{24}a^4-\tfrac{1}{720}a^6+\tfrac{1}{40320}a^8,\ \&c.$$

$$\text{Or } C=1-\frac{1}{1.2}a^2+\frac{1}{1.2.3.4}a^4-\frac{1}{1.2.3.4.5.6}a^6,\ \&c.$$

Corollary.

Again, becauſe the Radius leſſened by the Coſine, gives the verſed Sine of the ſame Arc.

If from 1 be taken $1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6$, &c. there will remain $\frac{1}{2}a^2-\frac{1}{24}a^4+\frac{1}{720}a^6$, &c. the Series for finding the verſed Sine from the Arc firſt given.

Alſo the Radius *plus*, the Coſine gives the verſed Sine of the Supplement; therefore 1 *plus* $1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6$, the Sum is $2-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{170}a^6$, &c. will be the Series for finding the verſed Sine of the Supplement from the Arc firſt given.

Thus having obtained the Sines and Coſines, the Tangents and Secants are eaſily had: For, becauſe the Triangles *Am*S, ACT, are ſimilar, it will be

As

As Am (the Cosine) : Sm (the right Sine) :: AC (Radius) : CT the Tangent.

Wherefore, because the Radius is supposed Unity, $\frac{\text{Right Sine}}{\text{Cosine}}$ = Tangent of the same Arc, *i. e.* the right Sine divided by its Cosine = Tangent of the same Arc. Hence the Tangent Series is easily found, for

$$\frac{a - \frac{1}{6}a^3 + \frac{1}{120}a^5, \text{ \&c. Right Sine}}{1 - \frac{1}{2}a^2 + \frac{1}{24}a^4, \text{ \&c. the Cosine}} = a + \frac{1}{3}a^3 + \frac{2}{15}a^5,$$

&c. the Tangent, as is evident from the following

OPERATION.

$$1 - \tfrac{1}{2}aa + \tfrac{1}{24}a^4 \text{ \&c.})\, a - \tfrac{1}{6}a^3 + \tfrac{1}{120}a^5 \text{ \&c.}\, (a + \tfrac{1}{3}a^3 + \tfrac{2}{15}a^5 \text{ \&c.}$$

$$a - \tfrac{1}{2}a^3 + \tfrac{1}{24}a^5, \text{ \&c.}$$

$$+ \tfrac{1}{3}a^3 - \tfrac{1}{30}a^5, \text{ \&c.}$$

$$+ \tfrac{1}{3}a^3 - \tfrac{1}{6}a^5, \text{ \&c.}$$

$$+ \tfrac{2}{15}a^5, \text{ \&c.}$$

Hence, if a be put for the Length of any Arc, the Correspondent Tangent will be, *viz.*

$$a + \frac{1}{3}a^3 + \frac{2}{15}a^5 + \frac{17}{315}a^7 + \frac{62}{2835}a^9 + \frac{1382}{155925}a^{11} + \frac{21844}{6081075}a^{13}, \text{ \&c.}$$

Again, because the Triangles CAT, GBA, are similar, it will be, as CT (Tangent) : AC (Radius) :: AB (the Radius) : BG the Cotangent.

COROLLARY.

Hence, the Radius is a mean Proportional between the Tangent and Co-tangent of an Arc; wherefore, because the Rad. is 1.

If $\frac{1}{a + \frac{1}{3}a^3 + \frac{2}{15}a^5, \text{ \&c.}} = \frac{1}{a} - \frac{1}{3}a - \frac{1}{45}a^3 - \frac{2}{945}a^5 - \frac{1}{4725}a^7 - \frac{2}{93555}a^9$, &c. will be the Series for finding the Co-tangent from an Arc first given.

Again

Again, becauſe the Triangles A*m*S, ACT, are ſimilar, it will be, as A*m* (Coſine) : AS (Rad.) : : AC (Rad.) : AT the Secant.

COROLLARY.

Hence, the Radius is a mean Proportional between the Secant and Coſine of an Arc.

Wherefore $\frac{1}{1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{2}{720}a^6, \text{ \&c. (Series for the Cosine)}} = 1+\frac{1}{2}a^2+\frac{1}{24}a^4+\frac{61}{720}a^6+\frac{277}{8064}a^8+\frac{50521}{3628800}a^{10}$, &c. will be the Series for finding the Secant, from the Length of an Arc firſt given.

Again, becauſe the Triangles *m*AS, ABG, are ſimilar, it will be, as *m*S (the right Sine) : AS (Rad.) : : AB (Rad.) : AG the Secant of the Compliment.

COROLLARY.

Hence the Radius is a mean Proportional between the right Sine and Co-ſecant of an Arc.

Wherefore, $\frac{1}{a-\frac{1}{6}a^3+\frac{1}{120}a^5, \text{ \&c.}} = \frac{1}{a}+\frac{1}{6}a+\frac{1}{360}a^3+\frac{31}{15120}a^5+\frac{127}{604800}a^7+\frac{73}{3421440}a^9$, &c. will be the Series for the Co-ſecant of the ſame Arc.

If $t=a+\frac{1}{3}a^3+\frac{2}{15}a^5+\frac{17}{315}a^7$, &c.

Then $a=t-\frac{1}{3}t^3+\frac{1}{5}t^5-\frac{1}{7}t^7+\frac{1}{9}t^9$, &c.

$$\left.\begin{array}{l} \text{For put } a=At+Bt^3+Ct^5, \text{ \&c.} \\ \text{Then will } \frac{1}{3}a^3=+\frac{1}{3}A^3t^3+A^2Bt^2, \text{ \&c.} \\ \text{And } \frac{2}{15}a^5= \qquad +\frac{2}{15}A^5t^5, \text{ \&c.} \end{array}\right\} =t.$$

And conſequently $At=t$, and $A=1$, alſo $B+\frac{1}{3}a^2=0$.

And $B=-\frac{1}{3}a^3=-\frac{1}{3}$; alſo $C+BA^2+\frac{2}{15}A^5=0$.

And $C=A^2B-\frac{2}{15}A^5=\frac{1}{3}-\frac{2}{15}=\frac{1}{5}$.

Wherefore

Wherefore $A=1$, $B=-\frac{1}{3}$, $C=\frac{1}{5}$, &c.
And consequently $a=t-\frac{1}{3}t^3+\frac{1}{5}t^5$, &c.

Hence follows the *Trigonometrical Series* in Order.

If a = Length of any given Arc, and x its Sine.

1. Then Arc $a=x+\frac{1.1}{2.3}x^3+\frac{1.3}{2.4.5}x^5+\frac{1.3.5}{2.4.6.7}x^7$, &c.

2. *Right Sine*, or $x=a-\frac{1}{6}a^3+\frac{1}{120}a^5-\frac{1}{2.3.4.5.6.7}a^7$, &c.

3. *Cosine*, $\sqrt{1-xx}=1-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6$, &c.

4. *Versed Sine* $=\frac{1}{2}aa-\frac{1}{24}a^4+\frac{1}{720}a^6$, &c.

5. *Versed Sine* Of the *Supplement* $=2-\frac{1}{2}a^2+\frac{1}{24}a^4-\frac{1}{720}a^6$, &c.

6. *Tangent* $=a+\frac{1}{3}a^3+\frac{2}{15}a^5+\frac{17}{315}a^7+\frac{62}{2835}a^9$, &c.

7. *Co-tangent* $=\frac{1}{a}-\frac{1}{3}a-\frac{1}{45}a^3-\frac{2}{945}a^5-\frac{1}{4725}a^7-\frac{2}{93555}a^9$, &c.

8. *Secant* $=1+\frac{1}{2}a^2+\frac{5}{24}a^4+\frac{61}{720}a^6+\frac{277}{8064}a^8+\frac{50524}{3628800}a^{10}$, &c.

9. *Co-secant* $=\frac{1}{a}+\frac{1}{6}a+\frac{1}{360}a^3+\frac{31}{15120}a^5+\frac{127}{604800}a^7+$ &c.

I have purposely omitted the Investigation of the three last Series; as also the Application of the Tangent, Secant, and Co-secant, *&c.* Series to the actual finding the Length of the Tangent, *&c.* from the Arc itself for Brevity Sake, it being very easy to any one who understands what went before.

Corollary.

If the Difference between Unity, and any greater Number, be called y, then the Log. of the Number,

$$1+y=y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\frac{1}{5}y^5, \text{ \&c.}$$

and if y be any Number less than Unity, then will the Logarithm of $1-y$, a Number less than Unity, be $=$ $-y-\frac{1}{2}y^2-\frac{1}{3}y^3-\frac{1}{4}y^4-\frac{1}{5}y^5$, &c.

I say, if the Radius be 1, and the Cosine of any Arch x, then the Sine will be $\sqrt{1-xx}$;

Then the Logarithm of $1+x=x-\frac{1}{2}x+\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5-\frac{1}{6}x^6$, &c.

And the Log. of $1-x=-x-\frac{1}{2}x-\frac{1}{3}x^3-\frac{1}{5}x^5-\frac{1}{6}x^6$, &c.

And the Log. of $\sqrt{1-xx}=\frac{1}{2}x^2-\frac{1}{4}x^4-\frac{1}{6}x^6$.

SCHOLIUM.

If the Radius or Tangent of 45° be 1. the Tangent of an Arch greater than 45° $=1+x$; and o one less $=1-x$; the Logarithm of the Tangent of the former Case will be $=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\frac{1}{5}x^5$, &c. and in the latter $=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\frac{1}{5}x^5$, &c.

Q. E. D.

Of the Measure *of* Ratios.

MEeting with a Collection of Mathematical Papers, in a large *Latin* Volume, I, after a Perusal of them, found them to have been the Study of an excellent Mathematician, the Reason of my conjecturing so, was, that the Method there laid down was excellently well handled, and in a very perspicuous Manner; I re-examined all Books of this Kind, and made the strictest Enquiry I could to know, whether the same had been communicated to us in our own Language, which to the best Information I could get, they were not, I immediately set myself to lay the same open to the meanest Capacity; and therefore, without Vanity or Presumption, I heartily offer it to the Public.

I shall here give the Reader a small Tract concerning *Ratios*. Now these Measures are Quantities of any Number soever, whose Magnitudes are analogous to the Magnitude of the Ratios; wherefore in this given *System*, the Measure is the same of the same Ratio, the double, of the Duplicate Ratio; the triple, of the Triplicate, *&c.* and in any Manner is the Measure likewise increased or diminished by the Composition, or Resolution of the increased or diminish'd Ratio. The Ratio of the Equality hath no Magnitude, because, being added or subtracted, induces no Mutation: The Ratios, which are called of a greater and lesser Inequality, hath contrary Affections of their Magnitudes, because they make the contrary in Composition and Resolution; wherefore, if the Measure of the Ratio, which the greater Term bears to the less, be affirmative, the Measure of the Ratio, which the less Term has to the greater, will be negative, but the Measure of the Ratio betwixt equal Terms will be of no Magnitude.

Moreover arises different *Systems* of Measures, as that determinate and immutable Analogy is shewn different Ways, which is between the Magnitudes of Ratios. From whence it appears, that infinite Systems may be exhibited

Therefore the Sum of all the Measures $M \times \frac{PQ}{AP}$ will be equal to the Measure sought of the proposed Ratio.

Q. E. I.

Corollary. 1.

The Terms AP, AQ, being so brought to Equality, that PQ may be the least Difference; $M \times \frac{PQ}{AP}$, or $M \times \frac{PQ}{AQ}$ will be equal to the Measure of the Ratio between AQ and AP to the Module M.

Corollary 2.

From whence the Module M is to the Measure of the Ratio between the Terms AQ, and AP, as either of the Terms AP, or AQ to the Difference of the Terms PQ.

Corollary 3.

Having given the Ratio between AC and AB, the Sum of all the $\frac{PQ}{AP}$ is given, and the Sum of all the $M \times \frac{PQ}{AP}$ is as M; wherefore the Measure of every Ratio is, as the Module of the System from which it is taken.

Corollary 4.

Therefore the Module in every System of Measures is always made equal to the Measure of a certain determinate and immutable Ratio, which therefore I shall call the *Ratio Modularis*.

Scholium 1.

Let the same be illustrated by an Example.

Let z be any determinate and permanent Quantity, and let x be the indeterminate, and variable by a continual

tinual Flux; then its Fluxion is $= \dot{x}$, and let the Measure of the Ratio be sought between $z+x$, and $z-x$. Let the Ratio be made equal to the Ratio between y and 1, and let y be denoted by AP, its Fluxion $\dot{y}$ by PQ; 1 by AB; then from the first Corollary will be gather'd, that the Fluxion of the sought Measure of the Ratio between y and 1, is $M \times \frac{\dot{y}}{y}$. Now for y put its Value $\frac{z+x}{z-x}$, and also for $\dot{y}$ the Fluxion of its Value $\frac{2z\dot{x}}{\overline{z-x}|^2}$, and the Fluxion of the Measure will be $2M \times \frac{z\dot{x}}{zz-xx}$, or $2M \times \frac{\dot{x}}{z - \frac{xx}{z}}$, or $2M$ into $\frac{\dot{x}}{z} + \frac{\dot{x}x^2}{z^3} + \frac{\dot{x}x^4}{z^5}$, &c. And consequently the Measure will become $2M$ in $\frac{x}{z} + \frac{xxx}{3z^3} + \frac{x^5}{5z^5}$, &c. whence the following

Corollary 5.

If the Sum of two Quantities be z, and Difference x, and $2M\frac{x}{z}$ be taken $= A$; $A\frac{xx}{zz} = B$; $B\frac{xx}{zz} = C$; $C\frac{xx}{zz} = D$, &c. the Measure of the Ratio, which the greater Quantity bears to the less, will be $\frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D +$ &c. $= \frac{B}{3} + \frac{C}{5} + \frac{D}{7}$, &c.

Scholium 2.

By a much like Process the Measure of the Ratio between $1+v$ and 1, will be M into $v - \frac{1}{2}v^2 + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5$,

$\frac{1}{5}v^5$ —, &c. wherefore, if that Measure be called m; $\frac{m}{M}$ will be $= v - \frac{1}{2}vv + \frac{1}{3}v^3 - \frac{1}{4}v^4 + \frac{1}{5}v^5$, &c. and therefore $\frac{mm}{MM} = vv - v^3 + \frac{11}{12}v^4 - \frac{5}{6}v^5$, &c. and likewise $\frac{m^3}{M^3} = v^3 - \frac{3}{2}v^4 + \frac{7}{4}v^5$, &c. Moreover, $\frac{m^4}{M^4} = v^4 - 2v^5$, &c. and lastly, $\frac{m^5}{M^5} = v^5$, &c.

Again, from the given Measure m, let us find the Ratio, as it is measured; by adding equal Things to equal Things, we shall have $\frac{m}{M} + \frac{mm}{2MM} = v * - \frac{1}{6}v^3 + \frac{5}{24}v^4 - \frac{13}{60}v^5$, &c. And again $\frac{m}{M} + \frac{mm}{2MM} + \frac{m^3}{6M^3} = v ** - \frac{1}{24}v^4 + \frac{3}{40}v^5$, &c. And again $\frac{m}{M} + \frac{mm}{2MM} + \frac{m^3}{6M^3} + \frac{m^4}{24M^4} = v *** - \frac{1}{120}v^5$, &c. And lastly, $\frac{m}{M} + \frac{mm}{2MM} + \frac{m^3}{6M^3} + \frac{m^4}{24M^4} + \frac{m^5}{120M^5} = v ****$ &c. that is.

$$\frac{m}{M} + \frac{mm}{2MM} + \frac{m^3}{6M^3} + \frac{m^4}{24M^4} + \frac{m^5}{120M^5} + \&c. = v;$$

wherefore the Ratio sought between $1 + v$, and 1 is as $1 + \frac{m}{M} + \frac{mm}{2MM} + \frac{m^3}{6M^3} + \frac{m^4}{24M^4} + \frac{m^5}{120M^5}$ + &c. to 1. Put $m = M$, or $\frac{m}{M} = 1$; and then the *Ratio Modularis* will be, as $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$, &c. to 1.

Likewise, if the Ratio be given between 1, and $1 - v$, the Measure of this Ratio will be M into $v + \frac{1}{2}v^2 + \frac{1}{3}v^3 + \frac{1}{4}v^4 + \frac{1}{5}v^5$, &c. and again, if the Measure of the Ratio

Ratio m be given, the Ratio will be, as 1 to $1 - \frac{m}{M} + \frac{mm}{2MM} - \frac{m^3}{6M^3} + \frac{m^4}{24M^4} - \frac{m^5}{120M^5} + \&c.$ put $m = M$, or $\frac{m}{M} = 1$. from thence the *Ratio Modularis* will be, as 1 to $1 - \frac{1}{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \&c.$ whence this

COROLLARY 6.

By denoting the Term R; if $\frac{1}{1}$ R be taken = A; $\frac{1}{2}$ A=B; $\frac{1}{3}$ B=C; $\frac{1}{4}$ C=D; $\frac{1}{5}$ D=E, &c. *ad infinitum*, and S be taken = R+A+B+C+D+E, *&c.* the *Ratio Modularis* will be that which is between the lesser Term denoted by R, and the greater Term S, which was now found. Or by denoting the Term S, if $\frac{1}{1}$ S be taken = A; $\frac{1}{2}$ A = B; $\frac{1}{3}$ B = C; $\frac{1}{4}$ C = D; $\frac{1}{5}$ D = E, &c. *ad infinitum*, and let R be taken = S—A+B—C+D—E+ *&c.* the *Ratio Modularis* will be that which is between the greater Term denoted by S, and the lesser Term R which we found. Moreover, the same Ratio is between 2.718281828459, *&c.* and 1. or between 1 and 0. 367879441171, *&c.*

SCHOLIUM 3.

If the lesser Terms are required, which may exhibit the same *Ratio Modularis*, that no lesser can be nearer than they; the Operation is to be investigated after the following Manner, let the greater Term 2. 718281828459, *&c.* be divided by the lesser 1, or likewise the greater 1 by 0.367879441171, *&c.* and again the lesser by the Remainder, and this again by the last Remainder, and so on, and the quotes will come out 2.1.2.

The Ratios greater than the Truth.		The Ratios less than the Truth.	
1	0×2	0	1
2	1	2	0
3	1×2	2	1×1
8	3	6	2
11	4×1	8	3×1
76	28	11	4
87	32×1	19	7×4
106	39	87	32
193	71×6	106	39×1
1264	465	1158	426
1457	536×1	1264	465×1
21768	8008	1457	536
23225	8544×1	2721	1001×8
25946	9545	23225	8544
49171	18089×10	25946	9545×1
&c.	&c.	&c.	&c.

1,1,4,1,1,6,1,1,8,1,1,10,1,1,12,1,1,14,1,1,16,1,1, &c.

These being found; the two Columns of Ratios are to be compleated, the one containing those that are greater than the true Ratio, the other containing those that are less than the true Ratio, by Beginning the Computation from the Ratios 1 to 0; 0 to 1, which are the farthest from the true Ratio; and thence subtracting to the other Ratios, which being continued, approach nearer to the Truth. Wherefore, let the Terms 1 and 0 be multiplied by the first Quote 2, and let the Factors 2 and 0 be set under the Terms 0 and 1, and by adding the *Ratio* will come out 2+0, to 0+1, or 2 to 1. Let the Terms of this be multiplied by the second Quotient 1, and the Factors 2 and 1 added to the Terms 1 and 0, and the Ratio will be as 2+1 to 1+0, or as 3 to 1. Let the Terms of this be multiplied by the third Quote 2, and the Factors 6 and 2 be added to the preceding Terms 2 and

2 and 1, and the Ratio will be as 8 to 3. Let the Terms of this be multiplied by the fourth Quote 1, and the Factors 8 and 3 added to the preceding Terms 3 and 1, and the Ratio will be 11 to 4. Let the Terms of this be multiplied by the fifth Quote 1, and the Factors 11 and 4 added to the preceding 8 and 3, the Ratio will be as 19 to 7. These Terms again let be multiplied by the sixth Quote 4, and the Factors 76 and 28 added to the preceding 11 and 4, and the Ratio will be found to be 87 to 32; and so on as far as you please. These being thus compleated, the Ratios greater than the true Ratio will be as 3 to 1, 11 to 4, 87 to 32, 193 to 71, 1457 to 536, 23225 to 8544, 49171 to 18089, &c. And the Ratios less than the true one will be 2 to 1, 8 to 3, 19 to 7, 106 to 39, 1264 to 465, 2721 to 1001, 25946 to 9545, &c. And these are the principal and primary Ratios, by which continually are approaching to the proposed Ratio.

But if the whole Series greater than the true Ratio be required, they may be given thus, that in the lesser Terms no designed Ratio greater than the true one, can approach nearer to the true Ratio; and likewise the whole Series of all the Ratios less than the true Ratio may be given thus, that in the lesser Terms no designed Ratio, less than the true Ratio, can approach to the true Ratio, so other secondary Ratios are to be inserted between the primary Ratios just now found; these are reckoned, when the Quotient exceeds Unity: Likewise they are found, by changing the Multiplication, which is made by the Quote, as above, in a continued Addition of Terms, as often as there are Unities in the Quotient.

Thus, the first Quote was 2, the Terms 1 and 0 are twice to be added to the Terms 0 and 1, and the Sums will give the Ratios 1 to 1. 2 to 1. These last Terms 2 and 1, because 1 was the second Quote, they are at once to be added to the Terms 1 and 0, and the Sums will give the Ratio 3 to 1. These Terms 3 and 1, because the third Quote was 2, they are to be added twice to the Terms 2 and 1, and the Sums will give the Ratios 5 to 2, 8 to 3. The last Terms 8 and 3, because the fourth Quote was 1, are at once to be added to the

Terms

Terms 8 and 3, and the Sum will give 19 to 7. Lastly, these Terms 19 and 7, because the sixth Quote was 4, they are four Times to be added to the Terms 11 and 4, and the Sums will give the Ratios 30 to 11, 49 to 18, 68 to 25, 87 to 32, and so might we proceed further.

Lastly, having performed the Operation, the whole Series of all the Ratios, greater than the true Ratio, will be 1 to 0, 3 to 1, 11 to 4, 30 to 11, 49 to 18, 68 to 25, 87 to 32, *&c.* and likewise the whole Series of all the Ratios, less than the true Ratio, will be 0 to 1, 1 to 1, 2 to 1, 5 to 2, 8 to 3, 19 to 7, *&c.*

The Ratios greater than the true Ratio.		The Ratios less than the true Ratio.	
1	0×2	0	1
2	1	1	0
3	1×2	1	1
8	3	1	0
11	4×1	2	1×1
19	7	3	1
30	11	5	2
19	7	3	1
49	18	8	3×1
19	7	11	4
68	25	19	7×4
19	7	87	32
87	32×1	106	39×1
&c.	&c.	&c.	&c.

The Usefulness of these Approximations extends itself to many others.

PROPOSITION II.

To construct Brigg's *Canon of* Logarithms.

Logarithms of composite Numbers are derived from *Logarithms* of the first composed Numbers, by Addition only,

only, and the Investigation of these may be had several Ways, as an Instance.

By the Fifth Corollary of the above Proposition, by writing 1 for M, the Logarithm of the Ratios are found between 126 and 125, 225 and 224, 2401 and 2400, 4375 and 4374, which may be called p, q, r, s, respectively; and the Logarithm of a decuple Ratio will be $239p + 90q - 63r + 103s$, or 2.302585092994, &c. wherefore, when *Brigg*'s Logarithm is a decuple Ratio, the Logarithm (*per* Cor. 3. Prop. 1.) may be made 2.302585092994, &c. as just above found to his *Module* 1. So *Brigg*'s Logarithm 1. of a decuple to his own *Module*, will be 0.43429448 1903, &c. Consequently therefore let the Value be put for M, and *Brigg*'s Logarithm of 7. 5. 3. will be $M \times \overline{202p + 76q - 53r + 87s}$, $M \times \overline{167p + 63q - 44r + 72s}$, $M \times \overline{114p + 43q - 30r + 49s}$. The Logarithm of the Number 2 is had by subtracting the Logarithm of 5 from the Logarithm of 10. And so are given also *Brigg*'s *Modulus*, and the Logarithms of all the Primes, which are less than 10.

The Logarithm of the first following Numbers, 11, 13, 17, 19, 23, &c. may be so computed. Then let the Facts from the Numbers next set on each Side in the first Proposition be sought; the Square of the first always exceeds the Fact by Unity. The Logarithm of the Ratio of the Square to the Fact by *Cor.* 5. *Prop.* 1. being found, the Logarithm of the same Fact may be added, which always will be composed of given Logarithms of the first Numbers, which are less in the first Proposition, and the half Sum will be the sought Logarithms of the first.

Corollary.

The Module of *Brigg*'s Logarithm is 0.43429448 1903, &c. its Reciprocal is 2.302585092994, &c.

Scholium.

After this Manner may the largest Table of Logarithms be compleated, such as *Brigg* or *Ulacque*. Or, suppose,

suppose we put l for the Logarithm as usual; then let $a+1$ be any proposed Number, and x its Logarithm to be found. Now, according to the Hypothesis, $x = l.\overline{a+1}$, which Equation may be called a general Canon. Let the Equation be made of a and y, any how composed, and combined with some other Numbers, any how by *Addition*, *Subtraction*, *Multiplication*, *Division* or *Extraction* of Roots. By the Assistance of this Equation, so taken at Pleasure, a will be exterminated from the general Canon, and the Equation, expressing the Relation between the indeterminate Numbers x, y will be had. The Fluxion of this Equation may easily be found, and its Integral or flowing Quantity expressed by an infinite Series, will give the known Value of x.

EXAMPLE 1.

Let a be assumed $= y$, then by the general Canon $x = l.\overline{1+y}$, whose Fluxion is $\dot{x} = \frac{\dot{y}}{1+y}$, and its Integral expressed by an infinite Series, gives

$$x = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \tfrac{1}{5}y^5 - \tfrac{1}{6}y^6 + \tfrac{1}{7}y^7, \text{ \&c.}$$

EXAMPLE 2.

Let y be assumed $= \frac{a}{a+2}$, whence $a+1 = \frac{1+y}{1-y}$; wherefore by the general Canon, $x = l.\frac{1+y}{1-y}$, whose Fluxion is $\dot{x} = \frac{2\dot{y}}{1-yy}$, and its Integral resolved into a Series, gives

$$x = 2 \times y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \tfrac{1}{7}y^7 + \tfrac{1}{9}y^9, \text{ \&c.}$$

Where the Number 2 is prefixed to the Series, is supposed to be multiplied into every Term of the Series.

Q. E. I.

LEMMA

LEMMA 1.

Let z be the Logarithm of any Fraction $\frac{b}{a+1}$, x the Logarithm of the Denominator $a+1$, x will be $= lb-z$; or, if z be the Logarithm of the Fraction $\frac{a+1}{b}$, x will be $= lb+z$.

LEMMA 2.

Let n be the Exponent of any Power of the Number b, $l.\ b^n$ will be $= n \times l.b.$ wherefore the Logarithm of the Number b^n, and the Exponent n being given, the Logarithm of b is given.

Let (as before) $a+1$ be the Number, whose Logarithm is x, to be found; and let b^n be the Product of the Numbers, greatest of which is less than $a+1$, and z the Logarithm of the Fraction $\frac{b}{a+1}$, that is, $z = l.\ \frac{b}{a+1}$, which Equation may be called a general Canon. Then for b, let the Quantity be taken from a, and let it be composed of any determinate Numbers whatever, and this Value of b, so taken at Pleasure, may be substituted in the Fraction $\frac{b}{a+1}$; whence it will be expressed by a, and given Numbers.

Let there be made any Equation between y and a, with Numbers taken at Pleasure, and by this a will be exterminated out of the general Canon, from whence is had the Equation expressing the Relation between the indeterminate ones z, y. The Fluxion of this Equation may easily be found, and its Integral expressed in an infinite Series, will give z the Logarithm of the Fraction $\frac{b}{a+1}$; and from z being found will be had (of the proposed Number $a+1$) the Log. of $x = l.b - z$. For, according to the Hy-

 pothesis,

pothesis, b^n is produced from Multiplication of the Numbers, whose greatest is less than $a+1$; and from the Hypothesis are given the Logarithms of all the Numbers, less than the proposed $a+1$; consequently the Logarithm of the Product, as b^n; and therefore (*per* Lem. 2.) the Logarithm of b is given.

Example 1.

Let b be taken $= a$, whence $z = l.\ \frac{a}{a+1}$; then (*per* Art. 2.) let y be taken at Pleasure $= 2a+1$, by this Equation a may be exterminated, and then will $z = l.\ \frac{y-1}{y+1}$, whose Fluxion is $\dot{z} = \frac{2\dot{y}}{yy-1}$, whose Integral expressed by a Series, gives

$$z = -2 \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7}, \text{ \&c. whence}$$

(*per* Lem. 1.) $x = lb + 2 \times \frac{1}{y} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \frac{9}{9y^9}$, &c.

Example 2.

Let b be made $= \sqrt{aa+2a}$, or $= \overline{aa+2a}|^{\frac{1}{2}}$, whence $z = l.\ \frac{\sqrt{aa+2a}}{a+1}$, let y be taken at Pleasure $= 2a+2a$, whence $z = l.\ \frac{1}{y}\sqrt{yy-4}$, whose Fluxion is $\dot{z} = 4\dot{y} \times \overline{y^3-4y}|^{-1}$, and its Integral is $\dot{z} = -2 \times \frac{1}{y^2} + \frac{2^2}{2y^4} + \frac{2^4}{3y^6} + \frac{2^6}{4y^8}$, &c. whence by Lem. 1. $x = l.\ b + 2 \times \frac{1}{y^2} + \frac{2^2}{2y^4} + \frac{2^4}{3y^6} + \frac{2^6}{4y^8} + \frac{2^8}{5y^{10}}$, &c.

EXAMPLE

Example 3.

Let b be made $= \overline{aa+2a}|^{\frac{1}{2}}$ as above; but now let y^2 be assumed $= 2aa + 4aa + 1$, if b and a be exterminated by these two Equations from the general Canon, z will be $= l\, \frac{\sqrt{yy-1}}{\sqrt{yy+1}} = l\, \frac{\overline{yy-1}|^{\frac{1}{2}}}{\overline{yy+1}|^{\frac{1}{2}}}$, whose Fluxion is $\dot{z} = 2y\dot{y} \times \overline{y^4-1}|^{-1}$, and its Integral is expressed by this Series $z = -\frac{1}{y^2} - \frac{1}{3y^6} - \frac{1}{5y^{10}} - \frac{1}{7y^{14}}$, &c. whence by Lem. 1.

$$x = l.\, b + \frac{1}{y^2} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}}, \ \&c.$$

It will not be amiss here to shew a particular Method of approximating in the Invention of Logarithms, which has no Occasion for any transcendental Methods, and is expeditious enough for making the Tables without much Trouble.

A New Method *of computing* Logarithms.

THIS Method is founded upon these Considerations.

1. That the Sum of the Logarithms of any two Numbers is the Logarithm of the Product of those two Numbers multiplied together.

2. That the Logarithm of Unit is nothing, and consequently, that nearer any Number is to unite, the nearer will its Logarithm be to 0. Thirdly, that the Product by Multiplication of two Numbers, whereof one is bigger, and the other less than Unit, is nearer to Unit, than that of the two Numbers which is on the same Side of Unit with itself. For Example, the two Numbers being $\frac{2}{3}$, and $\frac{4}{3}$, the Product $\frac{8}{9}$ is less than Unit, but nearer to it than $\frac{2}{3}$, which is also less than Unit. Upon these Considerations I found the present Approximation; which will be best explained by an Example. Let it

 therefore

therefore be propofed to find the Relation of the Logarithms of 2, and of 10. In order to this, I take two Fractions $\frac{128}{100}$, and $\frac{8}{10}$, *viz.* $\frac{2)^7}{10)^2}$, and $\frac{2)^3}{10)^1}$, whofe Numerators are Powers of 2, and their Denominators Powers of 10, one of them being bigger, and the other lefs than 1. Having fet thefe down in Decimal Fractions in the firft Column of the annex'd Table; againft them in the fecond Column I fet A and B for their Logarithms, expreffing by an Equation the Manner how they are compounded of the Logarithms of 2 and 10, for which I write $l2$ and $l10$; then multiplying the two Numbers in the firft Column together, I have a third Number 1.024, againft which I write C for its Logarithm, expreffing likewife by an Equation in what Manner C is formed of the foregoing Logarithms A and B. And in the fame Manner the Calculation is continued; only obferving this *Compendium*, that before I multiply the two laft Numbers already got in the Table, I confider what Power of one of them muft be ufed to bring the Product neareft to Unit that can be. This is found, after we have gone a little Way in the Table, only by dividing the Differences of the Numbers from Unit, one by the other, and taking the Quotient with the neareft, for the Index of the Power wanted. Thus the two laft Numbers in the Table being 0.8, and 1.024 their Differences from Unit are 0.200, and 0.024; therefore $\frac{0.200}{0.024}$ gives 9 for the Index; wherefore multiplying the ninth Power of 1.024 by 0.8, I have the next Number 0.990352031429, whofe Log. is $D=9C+B$. In feeking the Index in this Manner by Divifion of the Differences, the Quotient ought generally to be taken with the leaft; but in the prefent Cafe it happens to be the moft, becaufe, inftead of the Difference between 0.8 and 1, we ought ftrictly to have taken the Difference between the reciprocal 1.25 and 1, which would have given the Index 10; and that would be too big, becaufe the Product by that Means would have been bigger than 1, as 1.024 is. Whereas this Approximation requires, that the Numbers in the firft Column be alternately greater and lefs than 1, as may be feen in the Table.

The

The TABLE.

1,280000000000	A = 7*l*2—2*l*10 — — — — —	*l*2 > 0,28
0,800000000000	B = 3*l*2—*l*10 — — — — —	< 0,33
1,024000000000	C = B+A = 10*l*2—3*l*10 — — —	> 0,300
0,990352031429	D = 9C+B = 93*l*2—28*l*10 — — —	< 0,30107
1,004336277664	E = 2D+C = 196*l*2—59*l*10 — — —	> 0,301020
0,998959536107	F = 2E+D = 485*l*2—146*l*10 — — —	< 0,3010309
1,000162894165	G = 4F+E = 2136*l*2—643*l*10 — —	> 0,30102996
0,999936281874	H = 6G+F = 13301*l*2—4004*l*10 — — —	< 0,301029997
1,000035441215	I = 2H+G = 28738*l*2—8651*l*10 — —	> 0,3010299951
0,999971720830	K = I+H = 42039*l*2—12655*l*10 — —	< 0,3010299959
1,000007161046	L = K+I = 70777*l*2—21306*l*10 — —	> 0,30102999562
0,999993203514	M = 3L+K = 254370*l*2—76573*l*10 — —	< 0,30102999567
1,000000364511	N = M+L = 325147*l*2—97879*l*10 — —	> 0,3010299956635
0,999999764687	O = 18N+M = 6107016*l*2—1838395*l*10 —	< 0,3010299956640
Com.Ar. 235313		

0 = 364511O+235313N = 2302585825187*l*2—693147400972*l*10 > 0,30102999566398

When

When I have in this Manner continued the Calculation, till I have got the Numbers small enough, I suppose the last Logarithm to be equal to nothing. Which gives me an Equation, from which having got away the Letters by Means of the foregoing Equations, I have the Relation of the Logarithms proposed. In this Manner, if I suppose $G = 0$, I have $2136l2 - 643l10 = 0$, which gives the Logarithm of 2 true in seven Figures, and too big in the eight, which happens, because the Number corresponding with G is bigger than Unit.

There is another Expedient which renders this Calculation still shorter. It is founded upon this Consideration, that when x is very small, $\overline{1+x}|^n$ is very nearly $1+nx$. Hence, if $1+x$, and $1-z$, are the two last Numbers already got in the first Column of the Table and their Powers $\overline{1+x}|^m$, and $\overline{1-z}|^n$, are such as will make the Product $\overline{1+x}|^m \times \overline{1-z}|^n$ very near to Unit, m and n may be found thus; $\overline{1+x}|^m = 1+mx$, and $\overline{1-z}|^n = 1-nz$, and consequently $\overline{1+x}|^m \times \overline{1-z}|^n = 1+mx-nz-mnzx$, or (neglecting $mnzx$) $1+mx-nz$. Make this equal to 1, and we have $m:n::z:x::l\overline{1-z}:l\overline{1+x}$. Whence $xl\overline{1-z} + zl\overline{1+x} = 0$. To give an Example of the Application of this, let 1.024, and 0,990352 be the last Numbers in the Table, their Logarithms being C and D. Then we have $1.024 = 1+x$, and $0.990352 = 1-z$; and consequently $x = 0.024$, and $z = 0.009648$. Whence the Ratio $\frac{z}{x}$ in the least Numbers is $\frac{201}{500}$. So that for finding the Logarithms proposed, we may have $500\,D + 201\,C = 48510l2 - 14603l10$

$-14603 l10 = 0$, which gives $l2 = 0,3010307$, which is too big in the laſt Figure; but it is nearer the the Truth than what is got from the Logarithm F ſuppoſed equal to nothing. So that by this Means we have ſaved four Multiplications, which were neceſſary to find the Number 9989595, &c. Correſpondent to F, and which muſt have been had, if we would make the Logarithm true to the ſame Number of Places without this *Compendium*.

Sir Isaac Newton's *Differentials*, or *Method of Fluxions*, Illustrated.

THE principal Part of the Analytic Arc consists in any determinate Quantity, to be found in Numbers, but when the Nature of the Quantities and Numbers cannot be had, that all the Quantities are to be exhibited in Numbers accurately, we have Recourse to Approximations, that is, when the accurate Values of Quantities cannot be obtained mathematically, they are to be sought by those that are less distant by any given Difference.

What is handed down to us from the Ancients about this, is either particular, as their Method of reducing Quadratic Equations, or at least ill contriv'd for general Uses, as the Method of Exhaustions. *Vieta* was the first that shewed a general Method to reduce rational Equations, which only then were in Use. In this acquiesced all the Geometricians from his Time to Sir Isaac Newton's, who first brought to a Series, and afterwards applied the same to the Reduction of all Equations of all Kinds universally. And this Method proceeds from the first and last Ratios of Quantities, increasing or decreasing, that is, by Differences infinite small of coinciding Quantities. But the incomparable and sagacious Newton carried this Method still a great deal further, and taught us by what Method we must approximate to Quantities, which are determined by a regular Series of Terms, not as it is commonly made by an Equation. And thus he placed the Foundation of this Fluxionary *Calculus*, which proceeds by Differences of Quantities of every Magnitude, wherefore it is more universal than the Method

Method of Series. By Fluxions an Universal Doctrine of Approximations is deduced to the Solution of this *Problem. To find a Line that shall pass thro' any Number of given Points.* From this Solution, I say are found the Roots of any Equations whatsoever, and the Quantities, whose Relation may be expressed by other given Quantities, by no Equations hitherto known, and therefore I think that this Method of approximating is arrived to the highest Perfection.

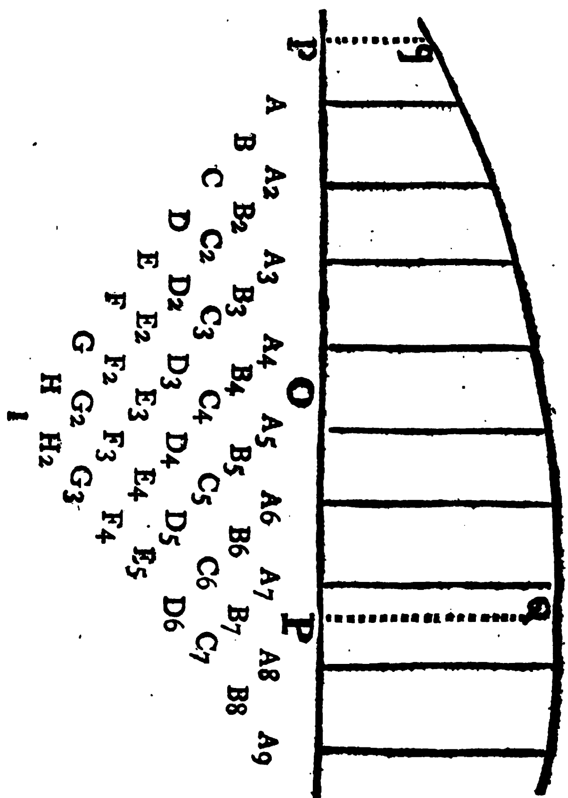
Our Author, in a Letter dated at *Oldenburgh, October* 24, 1676, makes mention of an expeditious Method of drawing a parabolic Curve thro' any Number of given Points, which he said he made use of, when the Simple Series would not suffice, and this Method he first published in *Lem.* 5. 3. *B. Princip.* and in his public Lectures about the same Time at *Cambridge*, whereby he shews a general Method to determine the Curves of what kind soever, which shall pass thro' as many Points, as their Nature will admit of, these Lectures were published under the Title *Arithmetica Universalis* in 1707, where he has illustrated the same by Examples in Conic Sections.

Archimedes, in the Method of Exhaustions; *Cavallerius*, in the Method of Indivisibles; and our *Wallis*, in his Arithmetic of Infinites, placed the Foundation of this Doctrine of a sought Quantity to be determined *per Locum*, which he obtained amongst the Terms in the given Series, but by what Method we must approximate to the Values of Quantities thus determined, no one hath taught us. This the sagacious and incomparable Sir ISAAC NEWTON was the first and only Person that brought it to Perfection; and from thence not a little was the universal Analysis enlarged. For before this Invention, those Arithmetical Problems only were to be solved, where the Relation of the Quantity sought, to others that are given, were defined by an Equation. Now the same may very expeditiously be solv'd, in which the Quantity sought takes the given Place amongst the Terms of the given Series; so the required Numbers are very accurately obtained by the Method of Fluxions, as by Extraction of Roots; these being had, it matters not how we approach to them. And a manifold Experience

Ee hath

hath taught us, that many Problems are difficult to be brought to Equations, except by Fluxions. As the Quadrature of the Circle so much spoken of, which, in my Opinion, *Wallis*, in his Arithmetic of Infinites, hath exhibited as perfect, as *Archimedes*, that of the *Parabola*.

To find a Parabolic Line which shall pass through the Extreams of the Ordinates, how many soever equidistant.



1. Let A. A2. A3. A4. A5. A6. A7. A8. A9. &c. denote the Ordinates equidistant on the Abscissæ in the given Angle.

Angle. Gather their Differences B. B2. B3. &c. and their Differences C. C2. C3.. &c. and their Differences D. D2. D3. &c. &c. And the Differences must be gathered by taking the former always from the latter, that is, by putting B=A2—A, B2=A3—A2, B3=A4—A3, B4=A5—A4, B5=A6—A5, &c. Then C=B2—B, C2=B3—B2, C3=B4—B3, C4=B5—B6, &c. And D=C2—C, D2=C3—C2, D3=C4—C3, D4=C5—C4, &c. And likewise are all the following Differences to be gathered. Or, let $\alpha, \beta, \gamma, \delta, \epsilon, \theta, \eta$, &c. be equal to A. A2. A3. A4. A5. A6. A7. &c. A will be $= \alpha$, $B = \beta - \alpha$, $C = \gamma - 2\beta + \alpha$, $D = \delta - 3\gamma + 3\beta - \alpha$, $E = \epsilon - 4\delta + 6\gamma - 4\beta + \alpha$, $F = \theta - 5\epsilon + 10\delta - 10\gamma + 5\beta - \alpha$, $G = \eta - 6\theta + 15\epsilon - 20\delta + 15\gamma - 6\beta + \alpha$, &c. In these Values, the Numeral Co-efficients of $\alpha, \beta, \gamma, \delta, \epsilon$, &c. are generated as in the integral Powers of the Binomial $\overline{1-z}|^{0}$, $\overline{1-z}|^{1}$, $\overline{1-z}|^{2}$, $\overline{1-z}|^{3}$, $\overline{1-z}|^{4}$, &c. by writing 1. 2. 3. 4. 5. &c. in the Series $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$ &c. successively for n. Let now PQ be any intermediate Ordinate. And AP its Distance from the first Ordinate A be called z, then will

$$PQ = A +$$

$$B \times \frac{z}{1} +$$

$$C \times \frac{z}{1} \times \frac{z-1}{2} +$$

$$D \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} +$$

$$E \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} +$$

$$F \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} +$$

$$G \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} \times \frac{z-5}{6} + \&c.$$

 Wherefore

Wherefore the Sign of z is changed, when PQ falls to the other Parts of the first Ordinate pq.

II. Now let A5 be the Ordinate in the middle of all; put $A = B4 + B5$, $B = D3 + D4$, $C = F2 + F3$, $D = H + H2$, &c. and $a = C4$, $b = E3$, $c = G2$, $d = I$. &c. that is, if A6 be $= \alpha$, $A7 = \beta$, $A8 = \gamma$, $A9 = \delta$, &c. $A4 = \kappa$, $A3 = \lambda$, $A2 = \mu$, $A = \nu$, &c.

Put $A = \alpha - \kappa$, $B = \beta - 2\alpha + 2\kappa - \lambda$, $C = \gamma - 4\beta + 5\alpha - 5\kappa + 4\lambda - \mu$, $D = \delta - 6\gamma + 14\beta - 14\alpha + 14\kappa - 14\lambda + 6\mu - \nu$, &c. $a = \alpha - 2A5 + \kappa$, $b = \beta - 4\alpha + 6A5 - 4\kappa + \lambda$, $c = \gamma - 6\beta + 15\alpha - 20A5 + 15\kappa - 6\lambda + \mu$, $d = \delta - 8\gamma + 28\beta - 56\alpha + 70A5 - 56\kappa + 28\lambda - 8\mu + \nu$, &c. and let A5P be called z, then will

$$PQ = A5 + \frac{Az + azz}{1.2} + \frac{2Bz + bzz}{1.2} \times \frac{zz - 1}{3.4} + \frac{3Cz + czz}{1.2} \times \frac{zz - 1}{3.4} \times \frac{zz - 4}{5.6} + \frac{4Dz + dzz}{1.2} \times \frac{zz - 1}{3.4} \times \frac{zz - 4}{5.6} \times \frac{zz - 9}{7.8} + \frac{5Ez + ezz}{1.2} \times \frac{zz - 1}{3.4} \times \frac{zz - 4}{5.6} \times \frac{zz - 9}{7.8} \times \frac{zz - 16}{9.10} + \text{\&c.}$$

III. Let A4, A5, be two Ordinates in the midst of all. Put $A = \frac{A4 + A5}{2}$, $B = \frac{C3 + C4}{2}$, $C = \frac{E2 + E3}{2}$, $D = \frac{G + G2}{2}$, &c. $a = B4$, $b = D3$, $c = F2$, $d = H$, &c. or, let $A5 = \alpha$, $A6 = \beta$, $A7 = \gamma$, $A8 = \delta$, &c. $A4 = \kappa$, $A3 = \lambda$, $A2 = \mu$, $A = \nu$, &c. then will $2A = \alpha + \kappa$, $2B = \beta - \alpha - \kappa + \lambda$, $2C = \gamma - 3\beta + 2\alpha + 2\kappa - 3\lambda + \mu$, $2D = \delta - 5\gamma + 9\beta - 5\alpha - 5\kappa + 9\lambda - 5\mu + \nu$, &c. And $a = \alpha - \kappa$, $b = \beta$ $- 3\alpha +$

$-3\alpha+3\varkappa-\lambda$, $c=\gamma-5\beta+10\alpha-10\varkappa+5\lambda-\mu$, $d=\delta-7\gamma+21\beta-35\alpha+35\varkappa-21\lambda+7\mu-\nu$, &c. And let O be the middle Point between A4, A5, and let OP be called z, and the Ordinate

$$PQ=\frac{A+az}{4^0}+$$
$$\frac{3B+bz}{4^1}\times\frac{4zz-1}{2.3}+$$
$$\frac{5C+cz}{4^2}\times\frac{4zz-1}{2.3}\times\frac{4zz-9}{4.5}+$$
$$\frac{7D+dz}{4^3}\times\frac{4zz-1}{2.3}\times\frac{4zz-9}{4.5}\times\frac{4zz-25}{6.7}+$$
$$\frac{9E+ez}{4^4}\times\frac{4zz-1}{2.3}\times\frac{4zz-9}{4.5}\times\frac{4zz-25}{6.7}$$
$$\times\frac{4zz-49}{8.9}+\&c.$$

In these two Cases z is negative, when the Ordinate PQ falls to the other Parts of the Beginning of the Abscissæ, and in all the three Cases, the common Difference of the Ordinates is put for Unity.

All the three Cases are very easily demonstrated by this Calculus. In the first Case for PQ, I write successively, α, β, γ, δ, ε, &c. and for z, 0. 1. 2. 3. 4. &c. which are the Lengths of the Abscissæ following in Order, and the Equations come out.

$\alpha=A$, $\beta=A+B$, $\gamma=A+2B+C$, $\delta=A+3B+3C+D$, $\varepsilon=A+4B+6C+4D+E$, &c.

$\beta-\alpha=B$, $\gamma-\beta=B+C$, $\delta-\gamma=B+2C+D$, $\varepsilon-\delta=B+3C+3D+E$, &c.

$\gamma-2\beta+\alpha=C$, $\delta-2\gamma+\beta=C+D$, $\varepsilon-2\delta+\gamma=C+2D+E$, &c.

$\delta-3\gamma+3\beta-\alpha=D$, $\varepsilon-3\delta+3\gamma-\beta=D+E$, &c.

$\varepsilon-4\delta+6\gamma-4\beta+\alpha=E$, &c.

These Equations, by taking their Differences, are resolved without any Trouble: And they give the same Values of A, B, C, D, &c. which are given before in the

the Solution, and after the same Manner are the other two Cases demonstrated.

Every one of these three Series will converge to the Value of the Ordinate PQ, when the Differences of the given Ordinates are of a just Magnitude; and when they do not converge, we must use other Methods. But for the present let us add a few Notes of the Use of this Proposition.

Let α, β, γ, δ, ε, θ, η, ι, κ, λ, &c. represent any equidistant Terms, whose Differences are very small; and the Relations, which they obtain among themselves, are very nearly defined by these following Equations, which arise by taking the Differences, and the Differences of those Differences continually, and by making them equal to nothing.

$$\alpha-\beta=0$$
$$\alpha-2\beta+\gamma=0$$
$$\alpha-3\beta+3\gamma-\delta=0$$
$$\alpha-4\beta+6\gamma-4\delta+\varepsilon=0$$
$$\alpha-5\beta+10\gamma-10\delta+5\varepsilon-\theta=0$$
$$\alpha-6\beta+15\gamma-20\delta+15\varepsilon-6\theta+\eta=0$$
$$\alpha-7\beta+21\gamma-35\delta+35\varepsilon-21\theta+7\eta-\iota=0$$
$$\alpha-8\beta+28\gamma-56\delta+70\varepsilon-56\theta+28\eta-8\iota+\kappa=0$$
$$\alpha-9\beta+36\gamma-84\delta+126\varepsilon-126\theta+84\eta-36\iota+9\kappa-\lambda=0.$$

This Table is to be reserved for Use, and to be consulted as often as is necessary; but that these Differences either obtain accurately, or approach to the Truth, when the Differences of the Terms are small, as appears from the Demonstration of the first Case of the Proposition. Let us assume any Series, as $\frac{1}{101}$, $\frac{1}{102}$, $\frac{1}{103}$, $\frac{1}{104}$, $\frac{1}{105}$, $\frac{1}{106}$, &c. and let the Term be sought which stands next before $\frac{1}{101}$; it is evident, that it is $\frac{1}{100}$, therefore we may here see that this Method will exhibit the same.

Let

Let a represent the Term fought, and it will be

	Equation	gives a
$\frac{1}{101}=\beta=$ 0099,0099,0099,0,	1	0099,0099,0099,0,
$\frac{1}{102}=\gamma=$ 0098,0392,1568,7,	2	0099,9805,8629,3,
$\frac{1}{103}=\delta=$ 0097,0873,7864,1,	3	0099,9994,3455,0,
$\frac{1}{104}=\epsilon=$ 0096,1538,4615,4,	4	0099,9999,7824,8,
$\frac{1}{105}=\theta=$ 0095,2380,9523,8	5	0099,9999,9895,8,
$\frac{1}{106}=\eta=$ 0094,3396,2264,2	6	0099,9999,9993,1,

It is evident, that this Method continually approaches nearer and nearer; if the Differences of the Terms had been lefs, the Values would have approached fooner to the Truth, and on the other hand, flower, when the Differences are greater; hence in the Numerical Table, if any Term be wanting, it may be inferted by this Method.

By this Method likewife come out the very fame Series, which ufed to be done by other Methods.

Suppofe $\overline{1+zz}|^{-1}$, the Ordinate of a Curve to be fquared, it is in the firft of the Ordinates in a regular Series $\overline{1+zz}|^{-1}$, $\overline{1+zz}|^{0}$, $\overline{1+zz}|^{1}$, $\overline{1+zz}|^{2}$, $\overline{1+zz}|^{3}$, &c. All which, except the firft, make their Areas, *viz.* z, $z+\frac{1}{3}z^3$, $z+\frac{2}{3}z^3+\frac{1}{5}z^5$, $z+\frac{3}{3}z^3+\frac{3}{5}z^5+\frac{1}{7}z^7$, &c. making a new Series, whofe firft Term will be the Area fought; which therefore will be found by putting a for it, and for the reft in Order, β, γ, δ, ϵ, &c. the firft Equation gives $a=z$, fecond, $a=z-\frac{1}{3}z^3$, the third, $a=z-\frac{2}{3}z^3+\frac{1}{5}z^5$, the fourth, $a=z-\frac{3}{3}z^3+\frac{3}{5}z^5-\frac{1}{7}z^7$, &c. Confequently the Area fought is univerfally, $z-\frac{1}{3}z^3+\frac{1}{5}z^5-\frac{1}{7}z^7+\frac{1}{9}z^9-\frac{1}{11}z^{11}$, &c. and this Series is the Arc to the Tangent z in a Circle whofe Radius is Unity.

Let, &c. e, d, c, b, a, P, α, β, γ, δ, ϵ, &c. the Series on both Sides going *ad infinitum*, when all the Terms are given except P in the midft of all, let $A=\alpha+a$, $B=\beta+b$, $C=\gamma+c$, $D=\delta+d$, $E=\epsilon+e$, &c. and it will be

P =

$$P = \frac{A}{2} + \frac{A-B}{6} + \frac{5A-8B+3C}{60} + \frac{7A-14B+9C-2D}{140} + \frac{42A-96B+81C-32D+5E}{1260} + \frac{66A-165B+165C-88D+25E-3F}{2772} + \frac{429A-1144B+1287C-832D+325E-72F+7G}{24024} + \&c.$$

This Series is investigated from the Equations, by taking every other, in which the Number of Terms is unequal; for their Difference will give the Terms in this Series, which may be produced at Pleasure.

Let $\overline{1+z}^{-1}$ be the Ordinate of the Hyperbola, and let its Area be sought, which is above the Abscissæ z, when it is Unity. This Ordinate is in the midst in the Series of these Ordinates, &c. $\overline{1+z}^{-5}$, $\overline{1+z}^{-4}$, $\overline{1+z}^{-3}$, $\overline{1+z}^{-2}$, $\overline{1+z}^{-1}$, $\overline{1+z}^{0}$, $\overline{1+z}^{1}$, $\overline{1+z}^{2}$, $\overline{1+z}^{3}$, &c. equidistant, hence continued *ad infinitum*. Consequently the Areas generated from these Ordinates will make the same Series, whose middle Term will be the Area sought, which will be obtained by the Series just now exhibited. When z is Unity, as in the present Case, the Area of Curves are, &c. $\frac{15}{64}$, $\frac{7}{24}$, $\frac{3}{8}$, $\frac{1}{2}$, and 1, $\frac{3}{2}$, $\frac{7}{3}$, $\frac{15}{4}$, &c. hence $A = 1 + \frac{1}{2} = \frac{3}{2}$, $B = \frac{3}{2} + \frac{3}{8} = \frac{15}{8}$, $C = \frac{7}{3} + \frac{7}{24} = \frac{21}{8}$, $D = \frac{15}{4} + \frac{15}{64} = \frac{255}{64}$, &c. these being substituted in the Series, comes out P, that is, the Area of the Hyperbola, $\frac{3}{4} - \frac{3}{48} + \frac{3}{480} - \frac{3}{2880} +$ &c. that is $\frac{3}{4} -$

$\frac{A}{4 \cdot 3}$

$\frac{A}{4 \cdot 3} - \frac{2B}{4 \cdot 5} - \frac{3C}{4 \cdot 7} - \frac{4D}{4 \cdot 9} - \frac{5E}{4 \cdot 11} -$ *&c.* where A, B, C, D, *&c.* represent the Terms in their Order from the Beginning after the *Newtonian* Method. I here put the *Calculus.*

TERMS.

Affirmatives.	Negatives.
7500,0000,0000,0000,0	0625,0000,0000,0000,0
62,5000,0000,0000,0	6,6964,2857,1428,5
7440,4761,9047,6	845,5086,5800,8
97,5586,9130,8	11,3818,4731,9
1,3390,4086,1	1585,7062,8
188,7745,5	22,5708,7
2,7085,0	3260,2
393,4	47,5
5,7	7
+ 7563,2539,3930,7494,1	— 0631,7821,3370,8041,1

Subtracting the Negative from the Affirmative, I get for the Area, (that is for the Hyperbolical Logarithm of 2) 6931,4718,0559,9453.

For the Construction of any of these Numerical Tables, the Series which follows is of great Use. Let e, d, c, b, a, α, β, γ, δ, ε, *&c.* represent the alternate Terms in the Series, being drawn out on each Side *ad infinitum.* Put $A = \alpha + a$, $B = \beta + b$, $C = \gamma + c$, $D = \delta + d$, $E = \varepsilon + e$, &c. and the Term between α and a will be

$$\frac{A}{2} +$$
$$\frac{1}{1} \times \frac{A - B}{2^4} +$$
$$\frac{1.3}{1.2} \times \frac{2A - 3B + C}{2^7} +$$
$$\frac{1.3.5}{1.2.3} \times \frac{5A - 9B + 5C - D}{2^{10}} +$$
$$\frac{1.3.5.7}{1.2.3.4} \times \frac{14A - 28B + 20C - 7D + E}{2^{13}} +$$
$$\frac{1.3.5.7.9}{1.2.3.4.5} \times \frac{42A - 90B + 75C - 35D + 9E - F}{2^{16}} +$$
$$\frac{1.3.5.7.9.11}{1.2.3.4.5.6} \times \frac{132A - 297B + 275C - 154D + 54E - 11F + G}{2^{19}} +$$
&c.

This Series follows from the third Cafe of this Propofition, by putting $z = 0$, thus are the Numeral Co-efficients of the Capitals produced, *Example*, in the fourth Term the Co-efficient of the laft Letter C, fave one, is 5; put $5 + 1 = n$, and the Numbers which come out from the Multiplication of the Terms, *viz.*

$$1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \&c.$$ will be 1, 6, 15, 20, &c. their Differences 5, 9, 5, are the Numbers fought, and confequently the Series may be produced at Pleafure.

Having given the Logarithms of 46, 48, 50, 52, 54, 56, 58, and 60, to find the Logarithm of 53, which is in the midft of all. Put $l52 + l54 = A = 3,4483,9710,34$, $l,50 + l,56 = B = 3,4471,5803,13$, $l,48 + l,58 = C = 3,4446,6923,08$, $l,46 + l,60 = D = 3,4409,0908,19$. Thefe Values being wrote in the Series, the four firft Terms will give 1,7242,2586,96, for the Logarithm of 53; and by the fame Method may we find any other intermediate Number.

Therefore in the Conftructing of the Tables it fufficeth, firft, to feek fome Terms in given Diftances, for the reft may be inferted after this Manner. For the Terms firft found

found are continually to be interplaced, until we come to the Terms, which are desired: After this Manner will be had the whole Table, from some few given Terms from the Beginning for the Foundation of our Operation. But it matters not, that the Terms, which we seek first, are all equidistant thro' the whole Table, for if we omit each other, where their Difference is the greatest, we may otherwise omit, two, three, twenty, or perhaps more Terms. But the Number of Terms being between those two that are given, which are omitted, must always be some one of the following, 1, 3, 7, 15, 31, 63, *&c.* so that we will insert them by this Series.

But for Praxis, the Terms may be collected into one Sum, as you may see here in the following Table, the first Expression is the first Term; the second, the Sum of the first and second; the third is the Sum of the first, second, and third, *&c.*

2	$\frac{A}{2}$
4	$\frac{9A-B}{16}$
6	$\frac{150A-25B+3C}{256}$
8	$\frac{1225A-245B+49C-5D}{2048}$
10	$\frac{39690A-8820B+2268C-305D+35E}{65536}$

So having given any alternate Terms, the intermediate will immediately be given by these Expressions, minding not the Nature of a particular Table. For these Rules are the same in all others. The Areas of Curves are nearly equal to the Areas of a Parabola, which passes through the Extreams of its Ordinates, but, because it would be too laborious to have Recourse always to the Parabola, I have composed the following Table, whereby the Areas are directly exhibited from the given Ordinates.

1	$\frac{A}{1}R$
3	$\frac{A+4B}{6}R$
5	$\frac{7A+32B+12C}{90}R$
7	$\frac{41A+216B+27C+272D}{840}R$
9	$\frac{989A+5888B-928C+10496D-4540E}{28350}R$
11	$\frac{16067A+106300B-48525C+272400D-260550E+427368F}{598752}R$

This Number of Ordinates is unequal, A is the Sum of the firſt and laſt; B of the ſecond, and laſt but one; C the third, and laſt but two, *&c.* until we come to that in the middle, which is repreſented by the laſt Letter in every Expreſſion. R is the Baſe, or Part of the Abſciſſæ, intercepted between the firſt and laſt Ordinate, the Expreſſions are the Areas contained between the Curve, Baſe, and Ordinates, from thence to the laſt. I have not conſtructed a Table for an even Number of Ordinates, becauſe the Area is defined more accurately *cæteris paribus*, of their unequal Number.

Let the Area be ſought, which is generated from the Ordinate $\overline{1+zz}|^{-1}$, and which lies above the Abſciſſæ z, when it is Unity: In $\overline{1+zz}|^{-1}$, for z write $\frac{0}{10}$, $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, $\frac{4}{10}$, $\frac{5}{10}$, $\frac{6}{10}$, $\frac{7}{10}$, $\frac{8}{10}$, $\frac{9}{10}$, $\frac{10}{10}$, and eleven Ordinates will come out 1, $\frac{100}{101}$, $\frac{25}{26}$, $\frac{100}{109}$, $\frac{25}{29}$, $\frac{4}{5}$, $\frac{25}{34}$, $\frac{100}{149}$, $\frac{25}{41}$, $\frac{100}{181}$, $\frac{1}{2}$. Hence is $A=1+\frac{1}{2}=\frac{3}{2}$, $B=\frac{100}{101}+\frac{100}{181}=\frac{28200}{18281}$, $C=\frac{25}{26}+\frac{25}{41}=\frac{1675}{1066}$, $D=\frac{100}{109}+\frac{100}{149}=\frac{25800}{16241}$, $E=\frac{25}{29}+\frac{25}{34}=\frac{1575}{986}$, $F=\frac{4}{5}$. Theſe

Values

Values being ſubſtituted in the laſt Expreſſion, and R be put for Unity, you will find the Area to be 785398187; this Number is exact in the 7th Figure, and in the 8th it exceeds the Truth by 2.

If eleven Ordinates do not give the Area exact enough, erect more, and conceive the Area to be divided into more Parts.

The Value of $\overline{1+Q}|^n$ may be expreſſed by any of the three following Series.

$$\overline{1+Q}|^n = 1 +$$
$$Q \times \frac{n}{1} +$$
$$Q^2 \times \frac{n}{1} \times \frac{n-1}{2} +$$
$$Q^3 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} +$$
$$Q^4 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$$
$$Q^5 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} + \&c.$$

Or $\overline{1+Q}|^n = 1 +$
$$R \times \frac{n}{1} +$$
$$R^2 \times \frac{n}{1} \times \frac{n+1}{2} +$$
$$R^3 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} +$$
$$R^4 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} +$$
$$R^5 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} + \&c.$$

by putting $R = \frac{1+Q}{Q}$. Or

$\overline{1+Q}|^n$

$$\overline{1+Q}^{n} = 1 +$$

$$\frac{2+\overline{n+1}\times Q}{\overline{1+Q}^{1}} \times Q \times \frac{n}{1.2} +$$

$$\frac{4+\overline{n+2}\times Q}{\overline{1+Q}^{2}} \times Q^{3} \times \frac{n}{1.2} \times \frac{nn-1}{3.4} +$$

$$\frac{6+\overline{n+3}\times Q}{\overline{1+Q}^{3}} \times Q^{5} \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} +$$

$$\frac{8+\overline{n+4}\times Q}{\overline{1+Q}^{4}} \times Q^{7} \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{3.6} \times$$

$$\frac{nn-9}{7.8} +$$

$$\frac{10+\overline{n+5}\times Q}{\overline{1+Q}^{5}} \times Q^{9} \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} \times$$

$$\frac{nn-9}{7.8} \times \frac{nn-16}{9.10} + \&c.$$

The two first Series are demonstrated by Case 1. of this Proposition. For if $\overline{1+Q}^{0}$, $\overline{1+Q}^{1}$, $\overline{1+Q}^{2}$, $\overline{1+Q}^{3}$, $\overline{1+Q}^{4}$, *&c.* represent so many equidistant Ordinates in the Parabola, $\overline{1+Q}^{n}$ will be its same Ordinate, whose Distance from $\overline{1+Q}^{0}$ is n. and thus is produced the first Series. And, if in any other Parabola $\overline{1+Q}^{0}$, $\overline{1+Q}^{-1}$, $\overline{1+Q}^{-2}$, $\overline{1+Q}^{-3}$, $\overline{1+Q}^{-4}$, *&c.* be equidistant Ordinates, $\overline{1+Q}^{n}$ will be the Ordinate in the same, whose Distance from $\overline{1+Q}^{0}$ is $-n$; thus will the second Series come out. Now let there be in the third Parabola, *&c.* $\overline{1+Q}^{-4}$, $\overline{1+Q}^{-3}$, $\overline{1+Q}^{-2}$, $\overline{1+Q}^{-1}$ $\overline{1+Q}^{0}$, $\overline{1+Q}^{1}$, $\overline{1+Q}^{2}$, $\overline{1+Q}^{3}$, $\overline{1+Q}^{4}$, *&c.* a Series of equidistant Ordinates, continued *ad infinitum*,

nitum, and its Ordinate will be $\overline{1+Q}|^{n}$, the Diſtance n removed from the middle Term $\overline{1+Q}|^{0}$. And ſo the third Series is produced by the ſecond Caſe of this Propoſition; the firſt fails, when n is an Integer and Affirmative; the ſecond, when n is an Integer and Negative; and the third fails in both Caſes. By the Help of which theſe numeral Roots are eaſily evolved into a Series; the third converges much ſooner; its ſecond Term may be exhibited for Correction, when the Extraction is made by the Repetition of the *Calculus*.

The ſagacious *Halley*, in his Method of conſtructing Logarithms from the firſt of theſe Series, demonſtrates *Mercator's* Series for the Quadrature of the Hyperbola. Let its Ordinates $\overline{1+z}|^{-1}$, or $\overline{1+z}|^{n-1}$, n being any Number infinitely ſmall, whence by the Methods of Squaring the Area, which lies above the *Abſciſſa* z, that is, the Logarithm of $1+z$ will be $\frac{\overline{1-z}|^{-1}}{n}$: But by the firſt Series $\overline{1+z}|^{n} = 1 + \frac{n}{1}z + \frac{n}{1} \times \frac{n-1}{2}z^2 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}z^3 +$ &c. and therefore in this preſent Caſe, where n is any infinite ſmall Number $\overline{1+z}|^{n} = 1 + \frac{n}{1}z - \frac{n}{2}z^2 + \frac{n}{3}z^3 - \frac{n}{4}z^4$, &c. which being ſubſtituted in the Value of the Area, it produces $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \frac{1}{5}z^5 - \frac{1}{6}z^6 + \frac{1}{7}z^7$, &c. which is *Mercator's* Series.

Likewiſe this Rule produces the ſame by the ſecond Series. Let the given Number be $1+z$, put $R = \frac{z}{1+z}$, and its Logarithm will be $R + \frac{1}{2}R^2 + \frac{1}{3}R^3 + \frac{1}{4}R^4 + \frac{1}{5}R^5 +$ &c.

The

The following Rule comes out by the third Series, let R represent any Number, put $z = \frac{\overline{R-1}|^2}{2R}$, and its Logarithm will be $\frac{RR-1}{2R} - \frac{1}{3}Az - \frac{2}{5}Bz - \frac{3}{7}Cz - \frac{4}{9}Dz - \frac{5}{11}Ez -$ &c. where A, B, C, D, E, *&c.* represent the Terms of the Series, as from the Beginning according to Sir ISAAC NEWTON's Method.

A METHOD *to find the Values of* Arithmetical Series, *how slow soever they converge.*

IN some Series the Sum of the Terms cannot be reckoned, but to very few Places of Figures, so that, except by a simple Addition of them, other Arts are not used. Now let any Series be proposed, all whose Terms are affected with the same Signs, and whose next continually tend to be equal amongst themselves, such as the following $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}$, *&c.* $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}$, *&c.* Gather the Sum of some Terms from the Beginning, and let α, β, γ, δ, ϵ, θ, *&c.* be added nearly in the nearest Numbers, let $r=\frac{\alpha\gamma-\beta\beta}{\alpha\beta-2\alpha\gamma+\beta\gamma}$, and let the Difference of the Quantities $\alpha\times\frac{\alpha+r\beta}{\alpha-\beta}$, $\alpha+\beta\times\frac{\beta+r\gamma}{\beta-\gamma}$, $\alpha+\beta+\gamma\times\frac{\gamma+r\delta}{\gamma-\delta}$, $\alpha+\beta+\gamma+\delta\times\frac{\delta+\epsilon}{\delta-\epsilon}$, $\alpha+\beta+\gamma+\delta+\epsilon\times\frac{\epsilon+r\theta}{\epsilon-\theta}$, *&c.* be a, b, c, d, e, &c. Then in the nearest Numbers, let $s=\frac{ac-bb}{ab-2ac+bc}$, and the Differences of $a\times\frac{a+sb}{a-b}$, $a+b\times\frac{b+sc}{b-c}$, $a+b+c\times\frac{c+sd}{c-d}$, $a+b+c+d\times\frac{d+se}{d-e}$, be A. B. C. D. *&c.* and let $t=\frac{AC-BB}{AB-2AC+BC}$, and so proceed as far as you please. Then will $\alpha+\beta+\gamma+\delta+\epsilon+$ &c. $=\alpha\times\frac{\alpha+r\beta}{\alpha-\beta}$

$\frac{\alpha+r\beta}{\alpha-\beta}+a\times\frac{a+sb}{a-b}+A\times\frac{A+tB}{A-B}+$ &c. and there is seldom any Occasion to carry on this new Series beyond the two first Terms.

As if the Value of this Series were desired, *viz.* of $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}+$ &c. gather the first 21 Terms, whose Sum I find to be 6813, 8410, 1885. The Terms next to be added are $\alpha=$,0005, 2854, 1226, $\beta=$,0004, 8309, 1787, $\gamma=$,0004, 4326, 2411, $\delta=$ 0004, 0816, 3265, &c. Hence let r be made $=1$ nearly, and $\alpha\times\frac{\alpha+r\beta}{\alpha-\beta}=$,0117, 6449, 6282, $a=-$,0000, 0017, 5096, $b=-$,0000, 0014, 7410, $c=-$,0000, 0012, 4986, &c. whence $s=\frac{5}{3}$ nearly, and $a\times\frac{a+sb}{a-b}$ $=-$, 0000, 0141, 8111, which I subtract from $\alpha\times\frac{\alpha+r\beta}{\alpha-\beta}$, because of its negative Sign, and there remains 0117, 6307, 8171; this added to the Sum first found 6813, 8410, 1885, gives 6931, 4718, 0056, for the Sum of the whole Series, which is exact in the ninth Decimal, but before these two Corrections, the Sum was exact in the first Figure only. If you have a Mind to pursue it further, it must be carried on to the following Approximations.

If the Terms of the Series have different Signs, they are to be added together, that all may have the same Signs, as in this Series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}$, &c. the Terms being added, it will be $\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\frac{2}{13.15}$ + &c. but here you must observe, that the Differences a, b, c, d, e, &c. as also A, B, C, D, &c. are to be gathered by subtracting the antecedent Quantities from the Consequent, and in all Series of this Nature, if p, q, r, represent the three Terms following in Order, p the first, q the second, r the third, and the Rectangle $\frac{p+r}{2}$

xq is not greater than pr, the Value of the Series will be infinitely great, but always of a finite Magnitude, when it happens to the contrary. This Rule fails us sometimes, when p, q, r, are but of a small Distance from the Beginning of the Series, and if they be amongst themselves a little remote from the Beginning, then the Rule is the most certain.

But to other Kinds of Series, other Rules must be applied; let there be a Series of regular Polygons inscribed in a Circle, the Radius being Unity.

H = 2, 0000, 0000, 0000, 000	4
G = 2, 8284, 2712, 4746, 190	8
F = 3, 0614, 6745, 8920, 718	16
E = 3, 1214, 4515, 2258, 051	32
D = 3, 1365, 4849, 0545, 938	64
C = 3, 1403, 3115, 6954, 752	128
B = 3, 1412, 7725, 0932, 772	256
A = 3, 1415, 1380, 1144, 299	512

Now let the last Polygon be called A, the last but one B, the last but two C, the rest in their backward Order, D, E, F, &c. and the Area sought of the Circle will be

$$A + \frac{A-B}{3} + \frac{4A-5B+C}{3.15} + \frac{64A-84B+21C-D}{3.15.63}$$
$$+ \frac{4096A-5440B+1428C-85D+E}{3.15.63.255} + \&c.$$ where if for A, B, C, D, E, &c. be wrote their proper Values, the four first Terms will give 3, 1415, 9265, 3589, 790, for the Area of the Circle. And this Series is general, no where depending on the Nature of the Circle; and it is applicable as often as the former Differences of the approximating Numbers are as the Quadruple of the latter. The Factors in the Denominators are whole Powers lessen'd by 4, which being given, the Co-efficients of the Letters in different Terms are formed by a continual Multiplication of $1, \frac{n}{3}, \frac{n-3}{15}, \frac{n-15}{63}, \frac{n-63}{255}$, &c. when the

laſt of the Factors in the Denominator muſt be ſubſtituted for *n*.

The laſt of the Quantities $x-1$, $2\sqrt[2]{x}-2$, $4\sqrt{x}-4$, $8\sqrt[8]{x}-8$, $\sqrt[16]{x}-16$, &c. is equal to the Logarithm of x. for x write 2, and by repeating the Extraction of Square Root, the Numbers will come out.

M =	1, 0000, 0000, 0000, 0000
L =	8284, 2712, 4746, 1901
I =	7568, 2864, 0010, 8843
H =	7240, 6186, 1322, 0613
G =	7083, 8051, 8838, 6214
F =	7007, 0875, 6931, 7337
E =	6969, 1430, 7308, 8294
D =	6950, 2734, 2438, 7611
C =	6940, 8641, 2851, 8363
B =	6936, 1658, 4759, 4014
A =	6933, 8182, 9699, 9493

Let the laſt of the Numbers be called A, the laſt but one be called B, and ſo backwards, and the Logarithm ſought will be $A + \frac{A-B}{1} + \frac{2A-3B+C}{1.3} + \frac{8A-14B+7C-D}{1.3.7} + \frac{64A-120B+70C-15D+E}{1.3.7.15} +$ &c. the five firſt Terms give 6931, 4718, 0559, 9457, for the Hyperbolical Logarithm of 2. And how this Series goes *ad infinitum* may be eaſily ſeen from what we have ſaid above; and likewiſe it is univerſal, no wiſe reſpecting the Proprieties of the Hyperbola.

This Differential Method likewiſe is extended to the Solution of Equations, and many other Uſes, of which I ſhall paſs over.

The

The Proportion *of* Mathematical Points *to each other.* *From the* Philosophical Transactions, *by* FRA. ROBERTS, Esq. F. R. S.

IT has heretofore passed for a current Maxim, that all Infinites are equal. Divines and Metaphysicians have not scrupled to ground many of their Arguments on that Foundation. The Position nevertheless is certainly erroneous, as Dr. *Halley* in Philosophical Transactions,
and has given diverse
hich are in a determi-
ıer, and some infinite-

ıfinitely small Quanti-
the followiug Propo-

N I.

ircles, and their Tan-
ı to the Diameters of

uch one another from
lraw the Tangent *paq*, *Fig. A.*
rom the Point *a* draw

...e, be equal to
R, and *ab*, the Diameter of ...e, be equal to S.

Let *db*, the Chord of the Ar... to *z*, and *fg*, the Chord of the Arch *fag*, ... and let the Abscissa *ak* be equal to *x*.

If the Line *mn* be supposed to move till ... incident with the Tangent *paq*, the Natur... will always give the following Equations.

$$zz = 4Rx - 4xx.$$
$$yy = 4Sx - 4xx.$$

Whe...

When the Line is arrived at the Tangent, z and y will become the two Points of Contact, and then $zz = 4Rx$, and $yy = 4Sx$ ($4xx$ being laid aside, as Heterogeneous to the rest of the Equation, by Reason of x being become infinitely small) Therefore

$$zz : yy :: 4Rx : 4Sx :: R.S.$$
$$\therefore \quad z : y :: \sqrt{R} : \overline{S}|^{\frac{1}{2}}. \quad \text{Q. E. D.}$$

Proposition II.

The Point of Contact between a Sphere and a Plane is infinitely greater than that between a Circle and a Tangent.

Let *a* be the Point of Contact between the Sphere *adqf*, *Fig.* B. and the Plane *bc*. (See *Fig.* B) About the Sphere describe the Cylinder *npgm*.

Draw *kh* to represent a Circle parallel to the Plane. Let the Circle be supposed to move, till it becomes Coincident with the Plane. The Cylindrical Surface *kghm* will always be equal (according to *Archimedes*) to the Spherical Surface *daf*.

Now when these Surfaces become infinitely small, one terminates in the Point of Contact, and the other in the Periphery of the Base of the Cylinder. Therefore the Point of Contact is equal to the Periphery of the Base of the Cylinder (equal to a Periphery, which has the same Diameter as the Sphere) and by Consequence is infinitely greater than any Point of Contact between a Circle and a Tangent. Q. E. D.

Propositition III.

The Points of Contact by Spheres of different Magnitude are to one another, as the Diameters of the Spheres.

For by the second Proposition the Points of Contact are equal to the Peripheries of such Diameter, whose Proportion is the same as the Diameters.

Q. E. D.

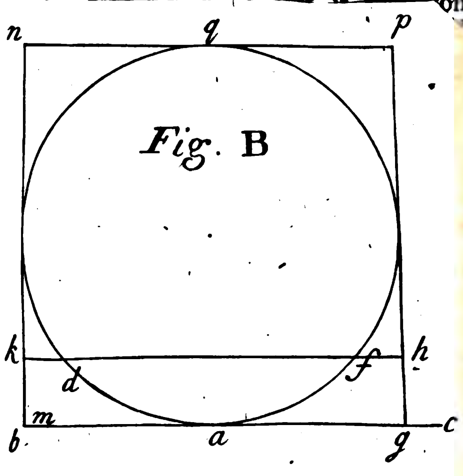
To

To find the Centre *of* Oscillation.

Definition.

Centre *of Oscillation* is a Point, wherein, if all the Gravity of a compound Pendulum be collected, every Oscillation will still be performed in the same … int of a Compound … Point of Suspension … lum, whose Os- … , as the Oscilla-

… ating Body, in … placed, it shall … ner in the same … ty as the whole



… *of a Body os- … ion may be the … nly.*

… of the proposed *Fig.* 10. … zon, in which … of Sus-pension being C. Imagine the Body … into innumerable small Prisms, all perpendicu… and consequently always parallel to the H… appear by the Motion of the Centre of G… the Plane ABD. And, because the Situatio… such Prism may be look'd upon as the Physical … placed at the Point z, in the same Plane ABD; co… quently the proposed Body may be reduced to the Phy… sical Plane ABD, consisting of such Particles *p*.

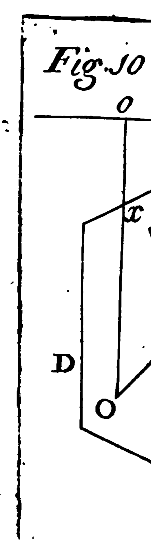
In

In this Plane, that the Point O may be found, whose proper Acceleration is not changed by the Actions of the other Particles, we may apply it to the Force of any single Particle *p*, placed in the Point *z*; of these Forces join'd together arises the absolute Motion of the whole Plane, by which the Motion of every Point is given.

But the Particle *p* is urged by the Force of its own Gravity; which, if the Cohesion of Parts be dissolved, in the least given Time, would produce the given Acceleration of the Motion in a perpendicular to the Horizon *zy*. To C*z* draw the Normal *yx*, and the Acceleration *zy* will be resolved into the Parts *zx* and *xy*. By Reason of the Rigidness of the Body the Force *zx* is taken away by the Resistance of the Point C; but by the other Force *xy*, is drawn the Space ABD, in a Ring about the Point C, and having drawn the Horizontal C*o*, and Perpendicular *zs*, it will be as $\frac{Cs}{Cz}$; because of the given Force of Gravity, and the Similiarity of Triangles *xyz*, and *s*C*z*. Therefore the Force of the Particle *p* to the Space to be moved ABD is, as $\frac{Cs}{Cz} \times p$.

To these Forces gathered into one Sum, let O be an invariable Point in a Line drawn at Pleasure, and to the Distance CO yet unknown; then the Force of the Particle *p* to move the Point O, as $\frac{Cz}{CO} \times \frac{Cs}{Cz} \times p$, that is, as $\frac{Cs}{CO} \times p$. and the Acceleration which *p* attributes to the same Point O will be, as $\frac{CO}{Cz} \times \frac{Cs}{Cz}$, wherefore the Force $\frac{Cs}{CO} \times p$, being applied to this Acceleration $\frac{CO \times Cs}{Czq}$, the Quotient will be $\frac{Czq}{COq} \times$ by the Particle *p*, which, if in the same Point O be supposed to be moved with the same Acceleration $\frac{CO \times Cs}{Czq}$, would produce the same Motion, which the Particle *p* produces

produces in the ſame Point O. Hence is reduced a Problem to the moſt known Theorem of Motions. For applying the Sum of the Forces $\frac{Cs}{CO} \times p$ to the Sum of the Particles $\frac{Czq:}{COq:} \times p$, the Quotient will be the abſolute Acceleration of the Point O, then having drawn the Perpendicular Oa, and this Acceleration being made equal to the given Acceleration $\frac{Co}{CO}$ of the Point O, the Diſtance CO will be given. For let $\frac{Co}{CO} = d$ (and according to the Method of Fluxions) $Cs \times p = \dot{M}$, and $Czq \times p = \dot{C}$; then becauſe CO being invariable, the Sum of all the Forces $\frac{Cs}{CO} \times p = \frac{M}{CO}$, and the Sum of all the Particles $\frac{Czq:}{COq:} \times p = \frac{C}{COq}$; whence the applicate Sum of the Moments to the Sum of the Bodies will be $\frac{M}{C} \times CO = d$; conſequently $CO = \frac{dC}{M}$, wherefore $\dot{C}$, and $\dot{M}$, being found by the Inverſe Method of Fluxions, CO will be given. Q. E. I.



Corollary.

From the Centre of Gravity G to the Horizontal Co draw the Perpendicular Gg, and let the Body ABC = A, then from a well known Law, the Centre of Gravity M will be $= Cg \times A$; whence $CO = \frac{dC}{Cg \times A}$.

Prop. 2. Theorem I.

The ſame being put as before, let the Point O be ſought in the right Line CG, paſſing through the Centre of Gravity G; then will O be the Centre of Oſcillation of the Body A.

For

Fig. 11. For (See *Fig.* 11.) In this Case $\frac{Co}{CO}$ is made $=$ $\frac{Cg}{CG} = d$; whence CO $=$ ($\frac{dC}{Cg \times A}$, by Cor. Prop. 1.) $\frac{C}{CG \times A}$. But A is given, and the Point C being given, the Quantity C and CG are also given: whence CO is given, whatsoever the Inclination of the oscillating Body be to the Horizon. Consequently, *per* Def. and Prob. 1. O is the Centre of Oscillation of the Body A.

Q. E. D.

PROP. 3. THEOREM 2.

The same being put as above, let D *be the Aggregate of* $Gz^2 \times p$. *Then will* $CO = CG + \frac{D}{CG \times A}$.

Fig. 12. To CG (See *Fig.* 12.) draw the Normal ZF, $\overline{CZ}^2 : = \overline{CG}^2 : + \overline{GZ}^2 : - 2CG \times GF$; namely F falling within C and G. But when F falls in CG produced, $\overline{Cz}^2$ will be $= \overline{CG}^2 : + \overline{Gz}^2 : + 2CG \times Gf$. Therefore C $=$ (Aggregate of all the $\overline{Cz}^2 : \times p =$) $=$ to the Aggregate of all the $\overline{CG}^2 : \times p + \overline{Gz}^2 : \times p - 2CG \times GF \times p + 2CG \times Gf \times p$. And because the Centre of Gravity of G, is the Aggregate of all the $2CG \times GF \times p =$ to the Aggregate of all the $2CG \times Gf \times p$. Wherefore C $=$ Aggregate of all the $\overline{CG}^2 : \times p + \overline{Gz}^2 : \times p = \overline{CG}^2 : \times A + D$. And by *Theorem* 1. $CO = \frac{C}{CG \times A}$. Therefore $CO = CG + \frac{D}{CG \times A}$. Q. E. D.

COROLLARY.

Hence is given the Parallelogram CG $\times$ GO. For $GO = \frac{D}{CG \times A}$. But A and D are given; wherefore $CG \times GO = \frac{D}{A}$.

PROP. 4.

PROP. 4. THEOREM 3.

The same being put as above, if in the Point O *be constituted the Physical Particle* $\frac{CG \times A}{CO}$, *which being agitated by its own Gravity, oscillates about the Point* C; *the Motion of the Space* ABC *will be every where alike, as if it were agitated by the Oscillation of the Body* A.

It is evident from the Nature of the Centre of Gravity, *per* Prob. 1. for $\frac{CG \times A}{CO}$ is the Aggregate of all the $\frac{\overline{Cz}^2 : \times p}{\overline{CO}^2 :} = \frac{C}{\overline{CO}^2}$.

PROP. 5. PROB. 2.

The Magnitude of any Body A, *the Centre of Gravity* G, *and the Point of Suspension* C *being given; to find its Centre of Oscillation* O.

It is done *per* Theor. 1. by finding the Quantity C, or *per* Theor. 2. by seeking the Quantity D.

SCHOLIUM.

To investigate the Calculus in a particular Case, the Quantity C and D must be chose, as the Nature of the proposed Figure suggests. Then either of them being given, the other will be given also by Equation (Prop. 3.) $C = \overline{CG}^2 : \times A + D$. Whence likewise will be given the Parallelogram $CG \times GO = \frac{D}{A}$ (*per* Cor. Prop. 3.) $= \frac{C}{A} - \overline{CG}^2$. By the Help of which, from the Centre of Gravity, and the Point of Suspension being given, the Centre of Oscillation is given by Division only. Wherefore in any Example, it will be the best to find first this Parallelogram, either by the *Computus* of D, or by the Quantity

Fig.

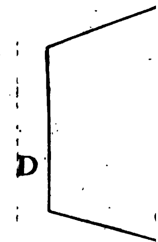
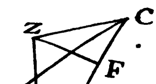


Fig. 12

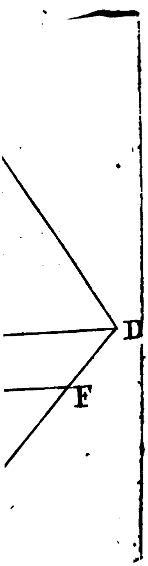


Quantity C, from a proper Assumption of the Centre of Suspension.

The Illustration of the same by an Example or two.

Example I.

Let there be a Pyramid ADC (See *Fig.* 13.) whose Base is the Parallelogram AD, and let the Motion of the Centre of Gravity in a Plane passing through the Vertex C, and the Diameter of the Base EF be parallel to the Side A.



Let the Vertex C be the Centre of Suspension; then *per* Prob. 1. the Figure may be reduced to a Physical Plane of an Isosceles Triangle CEF (See *Fig.* 14.) in which *ef*, parallel to EF, represents the Physical Line composed of the Particles *p*. Let $CH = a$, $HF = b$, and $Ch = x$; then *per* Property of the Figure *eb* will be $= \frac{bx}{a}$, and the Particle *p* situated at the Point *z* will be as x; or rather, bz being made $= v$; $\dot{v}\,\dot{x}$, will be the Base of the small Prism, and *p* will be as $\dot{v}\dot{x}x$; whence $\dot{C}$ will be $= \overline{Cz}^2 \times \dot{v}\dot{x}x = \dot{v}xx^3 + \dot{x}\dot{v}v^2x$. Consequently the Sum of all the $\overline{Cz}^2 \times p$, in the Line bz will be $\dot{v}xx^3 + \frac{\dot{x}xv^3}{3}$; and in the Line *ef* (for v by putting $\frac{bx}{a}$) that Sum will be $\frac{6ba^2+2b^3}{3a^3} \times \dot{x}x^4$. Whence again by taking the Fluent, and for x writing a, C will be $= \frac{6ba^2+2b^3}{15} \times a^2$, and $A = \frac{2baa}{3}$, and the Distance of the Centre of Gravity G from the Vertex C is $CG = \frac{3}{4}a$. Whence $\frac{C}{A} - \overline{CG}^2 = \frac{D}{A} = CG \times GO = \frac{3a^2+16b^2}{80}$.

Example

EXAMPLE 2.

Let the proposed Figure be a right Cone described by the Rotation of an Isosceles Triangle ECF about the Perpendicular CH.

Here again taking the Vertex C for the Centre of Suspension, and making CH$=a$, HE$=b$, C$h=x$, $hz=v$, as above; then will $\dot{p} = 2\dot{x}v \times \overline{\frac{bb}{aa}xx - vv}|^{\frac{1}{2}}$; whence $\dot{C} = 2v\dot{x} \times \overline{xx+vv} \times \overline{\frac{bb}{aa}xx - vv}|^{\frac{1}{2}}$. Let B be the Segment of a Circle described by the Diameter ef, which is near to the Absciss $hz = v$, and to the Ordinate $\overline{\frac{bb}{aa}xx - vv}|^{\frac{1}{2}}$; then the Sum will be of all the $\overline{Cz}^2 \times \dot{p}$ in the right Line $hz = 2\dot{x} \times \frac{4a^2+b^2}{4a^2} x^2 B - \frac{1}{3}\dot{x}v \times \overline{\frac{b^2}{a^2}x^2 - v^2}|^{\frac{3}{2}}$. And when $v = eh$, this Sum will be $2\dot{x} \times \frac{4a^2+b^2}{4a^2} x^2 B$; whose Double is $\frac{4a^2+b^2}{a^2} \dot{x}x^2 B$ is Part of C in the right Line ef, and the Area B, as x^2; therefore let B$=cx^2$, and that Part of C will be $\frac{4a^2+b^2}{a^2} \times c\dot{x}x^4$; whence by taking the Fluent, C $= \frac{4aa+bb}{5} \times ca^3$; and the Cone A $= \frac{4}{3}ca^3$, and CG $= \frac{3}{4}$A. Whence $\frac{C}{A} - \overline{CG}^2 := \frac{D}{A} = \frac{3a^2+12b^2}{30}$.

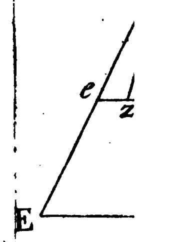
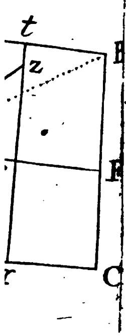


Fig. 34

And after this Method proceeds the *Calculus* in other Figures, when the Ratios Ch to he, and hz to p are more compounded.

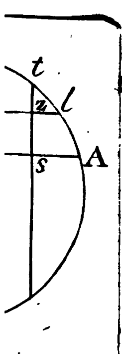
EXAMPLE

EXAMPLE 3.



That the Ratio of the Calculus of the Quantity D may be manifest; let there be a *Parallelepipedon*, perpendicular to the Horizon, and parallel to the Plane of the Motion of the Centre of Gravity ABD. Draw the Diameters EF and HI (See *Fig.* 15.) and let p be the Altitude of the small Prisms, and let tr be parallel to HI; and GF $= a$, GH $= b$, G$s = x$, and $sz = v$; then will $\dot{D} = \dot{v}xxx + \dot{x}vvv$. Whence the Part of D in the right Line tr will be $2bxx^2 + 2b^3x$; and again by taking the double of the Fluent, D will be $= \frac{4ba^3 + 4b^3a}{3}$; and A is $= 4ab$, whence $\frac{D}{A} = \frac{aa + bb}{3} = \frac{1}{12}$ DB squared.

EXAMPLE 4.



Let the last Example in a Sphere, whose greatest Circle is Btr, Diameter AB, and Centre G. Then having drawn the Lines as in the Scheme (See *Fig.* 16.) it is evident $\dot{D}$ will be $=$ Gsq : $\times p +$ Gmq : $\times p$; but the Sum of all the Gsq : $\times p$ in the right Line tr is Gsq, drawn into the Area of a Circle described by the Diameter tr. Likewise the Sum of all GMq : $\times p$ in the right Line ki is Gmq : $\times$ into the Area of the Circle described by the Diameter ki; when it immediately appears, that $\dot{D}$ is $=$ to four Times the Fluent of Gsq : into the Area of a Circle, whose Diameter is tr. Therefore let c be the Area of a Circle, the Square of whose Radius is 1. and let GA $= a$; and G$s = x$; then will $\dot{D} = 4\dot{x}xx \times caa - cxx = 4ca^2\dot{x}x^2 - 4c\dot{x}x^4$; whence taking the Fluent, and making $x = a$, D will be $= \frac{8}{15}ca^5$, and A $= \frac{4}{3}ca^3$; whence $\frac{D}{A} = \frac{2}{5}aa$.

By

By Reason of the Affinity of Solutions of Problems of this Nature I mean the Centre of Percussion) I will add the Solution of one Problem, only of the Centre of Percussion.

Definition.

Centre of Percussion is that Point of a Body in Motion, wherein all the Forces of that Body are united into one, or it is that Point, wherein the Stroke of the Body will be greatest.

The Centre of Percussion is the same as the Centre of Oscillation, if the striking Body revolves about a fixed Point; whence a Stick of a cylindrical Figure, supposing the Centre of Motion from the Hand, will strike the greatest Blow at a Distance about $\frac{2}{3}$ of its Length from the Hand.

The Centre of Percussion is the same as the Centre of Gravity, if all the Parts of the striking Body are carried by a parallel Motion, or move with the same Velocity.

Prop. 6. Prob. 3.

To find the Centre of Percussion of a Body whirl'd about a given Point, to wit, such a Point, that a Body impinging against it, and being loosed from the Point of Suspension, may neither incline to one Side or the other.

First, it is evident, that this Point must be sought in the Plane of the Motion of the Centre of Gravity. For if a Body be resolved into small Prisms (See *Fig.* 17.) as *Fig.* 17. Normals to that Plane, they will be carried about in a parallel Motion to themselves; whence the Moments of each Part of that Plane will be equal; consequently by the Resistance made in this Plane, no Point of the Body will be driven from it. Therefore let the Plane be AB, to which let the Body be reduced by Contraction of the small Prisms in the Particles p, situated in the Point x, as in Problem 1. In this Plane let C be the Centre of Rotation; or at least its Projection made by a perpendicular Line let fall on this Plane; and let Q be the Point sought. Thro' C draw Cξ at Pleasure, in which take two Points x and ξ, as xQ, and ξQ being drawn, the

the Angle CzQ may be obtuse, and the Angle CξQ acute; and in the Points z and ξ, let the Particles be p and π. Then to Cξ having drawn the Normals zr, and ξr; which are to one another, as Cz to Cξ, the absolute Velocities of the Particles p and π will be represented by them. And the Parts of these Velocities, which are in the Directions zQ, and ξQ, are taken away by the Resistance of the Point Q. And to Qz, and Qξ, draw the Normals CD, and Cd; and because of equal Angles $z\mathrm{CD} = rz\mathrm{Q}$, and $\xi \mathrm{C}d = r\xi\mathrm{Q}$, the other Parts of the Velocities in the Directions to the Perpendicular Qz, and Qξ, will be as zD, and ξd; whence having the Ratio of the Distances Qz, and Qξ, the Force of the Particles p and π will be to move the Space AB into contrary Parts, as $\mathrm{D}z \times z\mathrm{Q} \times p$, and $d\xi \times \xi\mathrm{Q} \times p$. And by the Conditions of the Problem, the Sums of such like contrary Forces must be equal among themselves.

By Reason the Angles being right at D and d, the Points D and d are at the Circumference of a Circle described by the Diameter CQ. Let the Centre of the Circle be E. Then having drawn Ez, and Eξ, meeting the Circle in F and I, f and i, $\mathrm{D}z \times z\mathrm{Q}$ will be $= \mathrm{F}z \times z\mathrm{I} = \overline{\mathrm{EF}}^2 : - \mathrm{E}z^2 : = \mathrm{EQ}^2 : - \mathrm{E}z^2 :$, and $d\xi \times \xi\mathrm{Q} = \mathrm{E}\xi^2 : - \mathrm{EQ}^2$. Wherefore the Sum of all the $\mathrm{EQ}q : \times p - \mathrm{E}zq : \times p$ will be $=$ to the Sum of all the $\mathrm{E}\xi q : \times \pi - \mathrm{EQ}q : \times \pi$; and transposing the Terms, the Sum of all the $\mathrm{EQ}q \times p + \pi =$ to $\mathrm{E}zq : \times p + \mathrm{E}\xi q : \times \pi$; that is, if p be put for the Particle p within the Circle, as well for the Particle π, without the Circle, the Sum of all the $\mathrm{EQ}q : \times p$ will be $=$ to the Sum of all the $\mathrm{E}zq : \times p$. To CQ draw the Normal zs; then will $\mathrm{E}zq : = \mathrm{C}zq : + \mathrm{EC}q - \mathrm{QC} \times \mathrm{C}s$. Which Value of E$zq$, being substituted for the same, and the Equation duly order'd, at length you will find the Sum of all the $\mathrm{CQ} \times \mathrm{C}s \times p =$ to the Sum of all $\mathrm{C}zq : \times p$, whence is

CQ =

$$CQ = \frac{\text{to the Sum of all the } Czq : \times p}{\text{Sum : of all the } Cs \times p}.$$

And the Sum of all $Czq : \times p$ is the ſame Quantity C in the Calculus of the Centre of Oſcillation. And if the Centre of Gravity be G, and to CQ be drawn the Normal Gg; and the ſame Body be called A, the Sum of all the $Cs \times p = Cg \times A$; whence is $CQ = \frac{C}{Cg \times A}$. Let the Centre of Oſcillation be O; then *per* Theorem 1. will $CO = \frac{C}{CG \times A}$. Whence is $Cg : CG :: CO : CQ$; wherefore, thro' O draw the Perpendicular, drawn to CO, will paſs through the Point Q.

Q. E. D. & I.

 Of

Of the Motion *of a* Musical String.

Lemma 1.

Fig. 18. LET ADFB, *and* AΔΦB, *be two Curves* (*See Fig.* 18.) *whereof this is the Relation* [...]*tween them.*
[...] *be Ordinates*
[...] *the Ordi-*
[...] *Curve may*
[...] *last Ratio*
[...] D, *as* CΔ

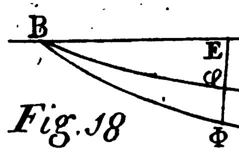


Fig. 18

Demonstration.

Draw the Ordinate *cδd* near to CD, and to D and Δ draw the Tangents D*t*, and Δθ, meeting the Ordinate *cd* in *t* and θ. Then, because *cδ* : *cd* :: CΔ : CD (*per* Hypothesis) and the Tangents produced one into another will meet the Axis in the same Point P. Whence by Reason of similar Triangles CDP, and *ct*P, CΔP, and *c*θP, it will be *c*θ : *ct* :: CΔ : CD (:: *cδ* : *cd per* Hyp.) :: δθ (= *c*θ—*cδ*) to *dt* (= *ct*—*cd*. But the Curvatures are in Δ and D, as the Contact of the Angle θΔδ, and *t*D*d*; and because δΔ and *d*D coincide with *c*C, these Angles are as their Subtenses δθ, and *dt*, that is (by the Analogy above found) as CΔ, and CD; wherefore, &c.

Q. E. D.

Lemma 2.

In any Article of its Vibration, the stretch'd String between the Points A *and* B, *takes the Form of every Curve*

Curve A$p\pi$B ... 9.
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πs, and ps in
Prin. Mechanics,
ticles pP, and Pπ, a
Force of the Thread's
of this Force, whereby the Particle pP alone is urged, will be to the String's Tension, as ct to tp, that is (by Reason of similar Triangles ctp, tpR) as tp, or Pp to Rt or PR. Wherefore, by Reason of the Tension's given Force, the absolute accelerating Power will be, as $\frac{Pp}{PR}$. But the generated Acceleration is in compound Ratio of the Ratios of the absolute Force directly, and of the Matter to be moved inversely; and the Matter to be moved is the Particle Pp: Wherefore the Acceleration is as $\frac{1}{PR}$, that is, as the Curvature in P. For the Curvature is reciprocally, as the Radius of the Osculatory Circle.

Q. E. D.

Problem I.

To define the Motion of a strech'd String.

In this Problem, and those that follow, I put the String to be moved by the least Space from the Axis of Motion, that the Increment of the Tension from its Length being augmented, also the Obliquity of the Radii of the Curve may safely be neglected; wherefore the String is extended between the Points A and B; and let the Point z be brought to the String to the Distance Cz from the Axis AB (See *Fig.* 20.). Then having moved the *Fig.* 20.

the String, by Reason of its Flexure in the Point C only, it will first begin to be moved (*per* Lem. 2.). And immediately the String being bent in the next Points ϕ and *d*: These Points likewise will begin to be moved, and then E and *e*, and so forwards. Likewise, by Reason the Flexure being great in C, that Point first will be moved with the greatest Velocity, and then augmenting the Curvature in the next Points D, E, &c. that continually will be accelerated swifter, and by the same Labour, the Curvature in C being diminished, that Point again will be accelerated the slower.

But that this may be manifest, the String always must take the Form of the Curve ACDEB, whose Curvature in any Point E, is, as its Distance from the Axis E*n*; likewise the Velocities of the Points C, D, E, &c. being made amongst themselves in the Ratio of the Distances from the Axis Cz, Dϑ, E*n*, &c. For in this Case, the Spaces CX, Dδ, Eϵ, &c. run over in the least Time, will be amongst themselves, as the Velocities, that is, as the Spaces Cz, Dϑ, &c. are to be run over; whence the other Spaces *xz*, $\delta\vartheta$, ϵn, &c. will be amongst themselves in the same Ratio. Likewise (*per* Lem. 2.) the Accelerations will be amongst themselves in the same Ratio. By which Means, the Ratio of the Velocities always being observed to be the same amongst themselves, as of their Spaces to be run over, all the Points will come together to the Axis, and will go together; wherefore the Curve ABDEB is rightly defined. Q. E. D.

Moreover, the two Curves ACDEB, and A$x\delta\epsilon$B, being compared between themselves (*per* Lem. 1.) the Curvatures will be in D and δ, and the Distances from the Axis Dϑ, and $\delta\vartheta$: Wherefore (*per* Lem. 2.) the Acceleration of any given Point in the String will be as its Distance from the Axis. When (*per Phil. Nat. Princ. Math. Sect.* 10. *Prop.* 51.) all the Vibrations, both the least and the greatest will be performed in the same Periodic Time, and the Motion of every Point will be like the Oscillation of a Funipendulous Body in the Cycloid.

Q. E. I.

COROLLARY

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as
]

Vibration.

Let the String be extended between the Points A and B (See *Fig.* 21.) by the Force of the Weight P, and let the Weight of the String be N, and Length L. Likewise let the String be made in the Position of AFpCB, and at the Middle Point C, erect the Normal CS = Radius of the Curvature in C, and meeting the Axis AB in D; and having taken the Point p near to C, draw the Normal pc, and the Tangent pt. *Fig.* 21.

Therefore, as in *Lemma* 2, it appears, that the absolute Force, whereby the Particle pC is accelerated, is to the Force of the Weight P, as ct to pt; *i. e.* as pC to CS. But the Weight P is to the Weight of the Particle pC, in a compound Ratio of the Ratios P to N, and N to the Weight of the Particle pC, or as L to pC, that is, as P×L to N×pC; wherefore these Ratios being compounded, the accelerating Force is to the Force of Gravity, as P×L to N×CS. Wherefore let a Pendulum be denoted by the Length CD (then *per* Princip. Math. Sect. X. Prob. 52.) the Periodic Time of the String will be to the Periodic Time of the Pendulum, as $\sqrt{N} \times CS$ to $\sqrt{P} \times L$. But (by the same Prop.) the Force of Gravity being given, the Lengths of Pendulums are in duplicate Ratio of the Periodic Times; whence it will be $\frac{N \times CS \times CD}{P \times L}$, or (for CS having wrote $\frac{aa}{CD}$ *per* Cor. Prob. 1.) $\frac{N \times aa}{P \times L}$ to the Length

of

of a Pendulum, whoſe Vibrations are Iſochronical to the Vibrations of the String.

To find the Line a, let the Abſciſſæ of the Curve be AE $= z$, and Ordinate EF $= x$, and the Curve AF $= v$, and CD $= b$. Then (*per* Cor. Prob. 1.) the Radius of the Curvature will be in F $= \frac{\dot{a}a}{\ddot{x}}$. But having given $\dot{v}$ the Radius of the Curvature $\frac{\dot{v}\dot{x}}{\ddot{z}}$. Whence $\frac{\dot{a}a}{\ddot{x}} = \frac{\dot{v}\dot{x}}{\ddot{z}}$; wherefore $aa\ddot{z} = \dot{v}x\ddot{x}$, and taking the Fluents $aa\dot{z} = \frac{\dot{v}xx}{2} - \frac{\dot{v}bb}{2} + \dot{v}aa$ (where the given Quantity $-\frac{\dot{v}bb}{2} + \dot{v}aa$ is added, that $\dot{z}$ may be made $= \dot{v}$ in the middle Point C). And hence the Calculus being ordered, $\dot{z}$ will be $=$ to $\frac{a^2\dot{x} - \frac{1}{2}b^2\dot{x} + \frac{1}{2}x^2\dot{x}}{\sqrt{a^2b^2 - a^2x^2 - \frac{1}{4}x^4 - \frac{1}{4}b^4 + \frac{1}{2}b^2x^2}}$. Now b and x vaniſh in reſpect of a, that the Curve may coincide with the Axis, and $\dot{z}$ will be made $= \frac{a\dot{x}}{\sqrt{bb - xx}}$. With the Centre C
Fig. 22. and Rad. CD$=b$, deſcribe a Quadrant DPE (See *Fig.* 22.) and making CQ $= x$, and erecting the Perpendicular QP, and the Arch DP being y, $\dot{y}$ will be $= \frac{b\dot{x}}{\sqrt{bb - xx}} = \frac{b}{a}\dot{z}$. Whence $y = \frac{b}{a}z$, and $z = \frac{a}{b}y$; and making $x = b =$ CD (in which Caſe likewiſe make $y =$ Arch of the Quadrant DPE, and $z =$ AD $= \frac{1}{2}$L) $\frac{1}{2}$L will be $= a \times \frac{\text{DE}}{\text{CD}}$, and $a = \text{L} \times \frac{\text{CD}}{2\text{DE}}$. Therefore let

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Periodic Times of *Pendulums* are in half Ratio of their Lengths.

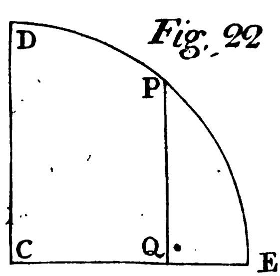
COROLLARY. I.

The Number of Vibrations of a String in the Time of one Vibration of a Pendulum D is $\frac{c}{d} \times \sqrt{\frac{p}{N}} \times \frac{D}{L}$.

COROLLARY 2.

Because $\frac{d}{c} \times \sqrt{\frac{1}{D}}$ is given, the Periodic Time of the String is, as $\sqrt{\frac{N}{P} \times L}$, and the Weight p being given, the Time is as $\sqrt{N \times L}$.

Fig. 22



Of the Laws *of* Centripetal Force.

A THEOREM.

IF a Body be moved in any Curve urged by Centripetal Force, that Force will be in any Point of the Curve, in a Compound Ratio of the direct Ratio of the Distance of the Body from the Centre of Force, and the Reciprocal Ratio of the Cube of the Perpendicular let fall on the Tangent in the ſame Point, and drawn into the Radius of the Curvature which the Curve there ſhall obtain.

Fig. 23. Let QAO be any Curve (See *Fig.* 23.) and let AO
leaſt Time, P*m* its Tangent,
f equal Curve, that is, the
ery coincides with the Arch
fall perdendicularly from the
let OM be drawn Parallel
And O*m* exhibits the Force,
A is urged towards S. The
des perpendicularly from the
hat is, the Force tending to-
Body to become moveable,
urve to the Arch AO, by
by it was firſt brought, and
ng towards S, whereby the
ve AO, as O*n* to O*m*, or,
angles, as SP to SA. But
Bodies brought into Circles
locities apply'd to the Radii,
per Cor. Theor. 4. *S. J. Newton's Princip.* But the Velocity is reciprocally as SP, or, directly as $\frac{1}{SP}$, where-

fore

fore the Squares of the Velocities will be ... $\frac{1}{}$

Therefore the Force as
Body may be moved in

$\frac{1}{SP^2 \times AR}$: But it has b

as the Force tending to
be moved in any equicu
ing towards S: But the

$\frac{1}{SP^2 \times AR}$; wherefore w

$\frac{SA}{SP^3 \times AR}$, the Force t

$\frac{SA}{SP^3 \times AR}$.

COROLLARY.

If the Curve QAO be a Circle (See *Fig.* 24.) the Centripetal Force tending towards S will be as $\frac{SA}{SP^3}$. *Fig.* 24.

Wherefore, if the Centripetal Force tend to the Point S, situated in the Circumference (*per* 32 E. 3.) the Angle PAS will be equal to the Angle AQS. Wherefore by Similar Triangles ASP, ASQ; it will be AQ : AS :: AS : SP : ; whence $SP = \frac{AS^2}{AQ}$, and $SP^3 = \frac{AS^6}{AQ^3}$.

Whence $\frac{SA}{S} = \frac{SA \times AQ^3}{} = \frac{AQ^3}{}$, that is, because AQ

AS5.

Let DA Fig. 25

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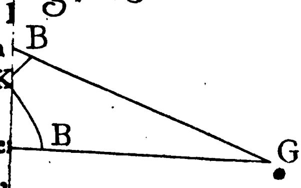
SA and K

SK :: (3.

being give

Lines SA,



AL = $\frac{1}{2}$ *Latus Rectum* = $\frac{1}{2}$L (*per Prop.* 6. *Partæ 4tæ Sect. Con. Milnij*).

Moreover, becauſe KA is to the Parallel SP, the Angle ASP=KAL=TOA, becauſe the Angle TAO is the Compliment of both at a right Angle: Wherefore KA : AL : : SA:SP. Whence SP $=\frac{L}{2}\times\frac{SA}{KA}$, and KA $=\frac{L}{2}\times\frac{SA}{SP}$.

Moreover, by Reaſon of Equi-angular Triangles, KM*k*, GPS, and OTA, SPA, it is

KM : K*k* :: GP : GS :: AP : SK
Alſo K*k* : AT :: SK : [illegible]
Alſo AT : AO :: AP

It will be KM : AO : AP^2 : SA^2 :: [illegible]
:: $SA^2-\frac{L^2\times SA^2}{4AK^2}$: SA^2 :: $4AK^2$ [illegible]
whence L^2 : $4AK^2$ :: (AO—KM : AC[illegible]
and its like Manner AR $=\frac{4AK^3}{L^2}$.
very ſame Reaſoning is found the Ra[illegible]
vature in the Hyperbola $=\frac{4AK^3}{L^2}=$ [illegible]

In the Parabola the Calculus is eaſie[illegible]
Fig. 26. of the given Subnormal (See *Fig.* 26. equal to AT, equal to the Fluxion of t[illegible] Triangles K*k*M, ATO, SPA, AKL [illegible], whence KM : K*k* :: AP : SA; likewiſe AT, or K*k* : AO :: AP : SA; whence KM : AO :: AP^2 : SA^2 :: SA^2-SP^2 : SA^2 :: whence it will be SP^2 : SA^2 :: AO — KM : AO :: AK : AR; and therefore AR $=\frac{SA^2\times AK}{SP^2}$; but AL = $\frac{1}{2}$ *Latus Rectum* = $\frac{1}{2}$ L; and AK : AL :: SA : SP; wherefore it will be $\frac{L}{2}\times\frac{SA}{AK}$ = SP,

$= SP$, and $SP^2 = \frac{L^2\times SA^2}{4AK^2}$; wherefore AR will be $=$ $\frac{4AK^3}{L^2}$; or because $AK = \frac{L\times SA}{2SP}$, AR will be $=$ $\frac{L\times SA^3}{2SP^3}$.

From hence arises a very easy Construction for determining the Radius of the Curvature in any Conic Section. For let AK be a Perpendicular in the Section meeting the Axis in K, (See *Fig.* 27.) From K on AK, let the Perpendicular HK be erected with AS pro- *Fig.* 27.
duced, meeting in H. From H let
on AH, be erected, AR will be t
Curvature; in the Parabola the Constr
more simple. For, because of the Natur
SA=SK; and the Angle AKH a right
Centre of a Circle passing thro' AKH; w
Radius of the Curvature by producing S
SA, and by erecting the Perpendicula
R will be the Centre of the osculating
rabola A.

The Centripetal Force tending to
Conic Section, in which a Body is mo
proportional to the Square of the Dist
$AR = \frac{L\times SA^3}{2SP^3}$, $\frac{SA}{SP^2\times AR}$ will be … $SP^3\times L\times SA^3$

$\frac{2}{L\times SA^2}$; that is, by Reason $\frac{2}{L}$ being given, the centripetal Force will be, as $\frac{1}{SA}$.

Let there be an Ellipsis BAD (See *Fig.* 28.) which *Fig.* 28. the right Line GE touches in A, and let SP passing thro' the Centre of the Ellipsis, and KA thro' the Contact be perpendicular on the Tangent. $SP\times KA$ will be $=$ to a fourth Part of the Figure of the Axis, or equal to the Square of the lesser Semiaxis $= BO\times DE$. For by Reason of equi-angular Triangles. GBO, GLA, GAK, GPS, and GDE.

Kk 2 SP:

$$SP : SG :: BO : GO$$
$$SG : DG :: BG : LG :: GO : GA$$
$$DG : DE :: GA : AK.$$

Whence $SP : DE :: BO : AK$, and $SP \times AK = DE \times BO = \frac{1}{2} L \times SB$.

Hence, if a moveable Body be moved in the Ellipsis by the centripetal Force tending to the Centre of the Ellipsis, that Force will be directly as the Distance; For $\frac{SP^3 \times 4AK^3}{L^2}$ = to a given Quantity. Because $SP \times AK$ is a given Quantity. Therefore, the Force, as $\frac{SA}{SP^3 \times AR}$, will be as SA the Distance.

Fig. 25. In *Fig.* 25. let fall the Perpendicular FI, from the other Focus F, on the Tangent; then by Reason of equi-angled Triangles SAP, FAI, it will be

$SA : SP : FA : FI = \frac{SP \times FA}{SA}$; whence will $SP \times FI = \frac{SP^2 \times FA}{SA}$ = Square of the lesser Semiaxis; whence, if the greater Axis be called b, and the lesser $2d$; then will $SP^2 = \frac{d^2 AS}{b - SA}$, and $SP = \frac{d SA^{\frac{1}{2}}}{\sqrt{b - SA}}$.

But in the Hyperbola $SP = \frac{d SA^{\frac{1}{2}}}{\sqrt{b + SA}}$.

In the Parabola, $SP = \sqrt{dSA}$; its *Latus Rectum* being put $= 4d$.

Because $TA^2 : TO^2 :: AP^2 : SP^2 :: SA^2 - SP^2 : SP^2 :: SA^2 - \frac{d^2 SA}{b - SA} : \frac{d^2 SA}{b - SA} :: SA - \frac{d^2}{b - SA} : \frac{d^2}{b - SA} :: bSA - SA^2 - d^2 : d^2$; it will be

$\sqrt{bSA - SA^2 - d^2} : d :: TA : TO$, when $TA = SA$, TO will be $= \frac{dSA}{\sqrt{bSA - SA^2 - d^2}}$.

Now

Now let QAO be any Curve, whose least Arch let be AO (See *Fig*. 29.) and AP, O*p*, Tangents in the Points A and O; Let SP, S*p* ... Tangents, AR the Radius ... *Fig*. 29.

$\frac{SA \times TA}{fP} = AR$, and the Per...

let be SP, S*p*. For by Reaso...
it is $fP : AO :: PA : RA$, a...
whence *ex æquo*, it will b...

RA; but $fP = S\dot{P}$, wherefo...

Hence, if the Distance SA ...
and be divided by the Fluxio...
Radius of the Curvature will ...
rem, the Curvature in Radial Curves is easily determined.

EXAMPLE.

Let AQ be a Nautical Spiral, seeing the Angle SAP is given, the Ratio SA to SP will be given also, let that Ratio be as a to b, SP will be $= \frac{bSA}{a}$, and $S\dot{P} = \frac{bS\dot{A}}{a}$, and $AR = \frac{SAS\dot{A}}{S\dot{P}} = \frac{aSA}{b}$; whence it will easily appear, that the Evolute of the Nautical Spiral is the same in any other Position, seeing $AR = \frac{SAS\dot{A}}{S\dot{P}}$, $\frac{SA}{SP^3 \times AR}$ will be $= \frac{S\dot{P}}{\quad}$. And again from the given Relation SA

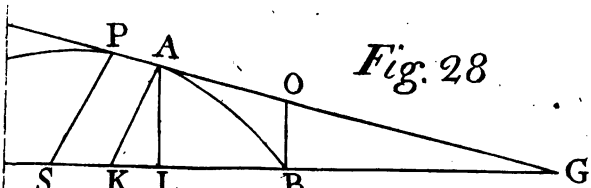


Fig. 28

Latus Rectum = 2R; and let V*a*Q be another Curve so related to this, that the Angle VSA be perpetually proportional to the Angle VS*a*, and let S*a*=SA. The Law of centripetal Force tending to S is sought, whereby a Body may be moved in the Curve V*a*Q.

Seeing the Angle VSA is to VS*a* in a given Ratio; the Increments of these Angles will be in the same Ratio; and let this Ratio be as *m* to *n*, whence *ot* will be $= \frac{n \times OT}{m}$. But $OT = \frac{d\dot{SA}}{\sqrt{bSA - SA^2 - d^2}}$, whence $ot = \frac{nd\dot{SA}}{m\sqrt{bSA - SA^2 - d^2}}$. And seeing $SA^2 + SP^2 : SP^2 :: ta^2 + ot^2 : ot^2 :: \dot{SA}^2 + \frac{n^2 d^2 \dot{SA}^2}{m^2\, bSA - SA^2 - d^2} : \frac{n^2 d^2 \dot{S}^2}{m^2\, bSA - SA^2 - d^2} :: 1 + \frac{n^2 d^2}{m^2 \times bSA - SA^2 - d^2} : \frac{n^2 d^2}{m^2 \times bSA - SA^2 - d^2} :: m^2 bSA - m^2 SA^2 - m^2 d^2 + n^2 d^2 : n^2 d^2$; whence it will be $\sqrt{m^2 bSA - m^2 SA^2 - m^2 d^2 + n^2 d^2} : nd :: SA : SP$, and $SP = \frac{ndSA}{\sqrt{m^2 bSA - m^2 SA^2 - m^2 d^2 + n^2 d^2}}$. That the Fluxion of which may be had, let *x* be wrote for $m^2 bSA - m^2 SA^2 - m^2 d^2 + n^2 d^2$; and SP will be $= \frac{ndSA}{\sqrt{x}}$, and $SP^3 = \frac{n^3 d^3 SA^3}{x^{\frac{3}{2}}}$; and $\dot{x} = m^2 b\dot{SA} - 2m^2 SA\dot{SA}$, and $\dot{SP} = nd\dot{SA} \times x^{-\frac{1}{2}} \frac{nASA\dot{x}}{x^{\frac{3}{2}}}$, and by reducing the Parts to the same Denominator; $\dot{SP}$ will be $= \frac{nd\dot{SA}x - \frac{1}{2}ndSA\dot{x}}{x^{\frac{3}{2}}}$, and in the Numerator, in the Place of *x* and $\dot{x}$, by putting their Values, and ordering the same, is made $\dot{SP} = \frac{nd\dot{SA} \times \frac{1}{2}m^2 bSA - m^2 d^2 + n^2 d^2}{x^{\frac{3}{2}}}$,

whence

whence $\frac{\dot{SP}}{SP^3 \times \dot{SA}} = \frac{\frac{1}{2}m^2 b SA - m^2 d^2 + n^2 d^2}{n^2 d^2 SA^3}$. But $\frac{\dot{SP}}{SP^3 \times \dot{SA}}$ is, as the Centripetal Force; wherefore the Force will be as $\frac{\frac{1}{2}m^2 b SA - m^2 d^2 + n^2 d^2}{n^2 d^2 SA^3}$; or (because $n^2 d^2$ will be in the Denominator) the Force will be as $\frac{\frac{1}{2}m^2 b SA - m^2 d^2 + n^2 d^2}{SA^3}$, or instead of d^2 by putting $\frac{bR}{2}$, the Force will be as $\frac{\frac{1}{2}m^2 b SA - \frac{1}{2}m^2 bR + \frac{1}{2}n^2 bR}{SA^3}$; or, (because $\frac{b}{2}$ is given) as $\frac{m^2 SA - Rm^2 + Rn^2}{SA^3} = \frac{m^2}{SA^2} + \frac{Rn^2 - Rm^2}{SA^3}$. All which exactly coincide with those which Sir *Isaac Newton* delivered in Prop. 44. of his *Principia* concerning the Centripetal Force of a Body moving in the same Curve.

Forasmuch as the Centripetal Force tending to the Point S, which being urged, a Body may be moved in the Curve, is always as $\frac{\dot{SP}}{SP^3 \times \dot{SA}}$; hence from the given Law of Centripetal Force the Relation of SA to SP may be found; as therefore by the inverse Method of Tangents the Curve may be exhibited, which might be described by any given Centripetal Force.

EXAMPLE.

Let the Force be reciprocally as any Power m of the Distance, that is, let $\frac{\dot{SP}}{SP^3 \times \dot{SA}} = \frac{b}{a^2 SA^m}$; $\frac{\dot{SP}}{SP^3}$ will be $= \frac{b\dot{SA}}{a^2 SA^m}$, and taking the Fluents of these

Fluxions

Fluxions $\frac{1}{2}SP^{-2}$ will be $= \frac{bSA^{1-m} \mp e}{\overline{m-1} \times a^2}$, whence will

$$\frac{\frac{m-1}{2} \times a^2}{bSA^{1-m} \pm e} = SP^2,$$ and by multiplying both Numerator and Denominator of the Fraction by SA^{m-1}; and, instead of $\frac{m-1}{2} a^2$ put d^2 is $\frac{d^2SA^{m-1}}{b \mp eSA^{m-1}} = SP^2$;

wherefore $SP = \frac{d\sqrt{SA^{m-1}}}{\sqrt{b+eSA^{m-1}}}$.

But if e be a constant Quantity SP $\frac{\sqrt{SA^{m-1}}}{\sqrt{b}}$ will be equal to nothing.

Wherefore, if the Force be reciprocally as the Square of the Distance; SP may be put $= \frac{\sqrt{d^2SA}}{\sqrt{b}}$, and the Curve will be a Parabola, whose *Latus Rectum* is $\frac{4d^2}{b}$, or you may put $SP = d \times \frac{\sqrt{SA}}{\sqrt{b-SA}}$, and the Curve will be an Ellipsis; or finally, you may put SP $= d \times \frac{\sqrt{SA}}{\sqrt{b \times SA}}$, and the Curve appears an Hyperbola.

If the Force be reciprocally, as the Cube of the Distance, it may be supposed, that $SP = \frac{dSA}{b}$, and the Curve be a Nautical Spiral, or $SP = \frac{dSA}{\sqrt{b-eSA}}$, and the Curve will be the same, whose Construction Sir *Isaac Newton* sought from the Sector of the Hyperbola, or may

may be as $SP = \frac{dSA}{\sqrt{b + eSA^2}}$, and Sir *Isaac* gives the Construction of the same Curve by Elliptical Sectors, *Cor.* 3. *Prop*

If the Ce

tance; the R

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la; for SP is

Logarithm o

Now, let [illegible] be moved in the Curve QAO, (See *Fig.* 23.) by the urging Centripetal Force tending *Fig.* 23. to S; and let the Celerity of a Body in A be called C; and the Celerity with which a Body, with the same urging Centripetal Force, in the same Distance, moved in a Circle be called c. It appears from the first Theorem; that if SA exhibit the Centripetal Force tending to S; the Centripetal Force, tending to R, will be exhibited by SP, which being urged, the Body with the Celerity C, will describe a Circle, whose Radius is AR; and the Centripetal Forces of Bodies describing Circles are as the Squares of the Velocities applicate to the Radii of the Circles; wherefore it will be $SP : SA :: \frac{C^2}{AR} : \frac{c^2}{SA}$; whence it will be $SP \times AR : SA^2 :: C^2 : c^2$, and $C : c :: \sqrt{SP \times AR} : SA$.

If SP coincide with SA, as in the Vertices of the Figures, it will be $C : c :: \sqrt{AR} : \sqrt{SA}$: But if the Curve AR be a Conic Section, the Radius of the Curvature in its Vertex is equal to half the *Latus Rectum* $= \frac{1}{2} L$. And in like Manner, the Velocity of a Body in the Vertex of the Section is to the Velocity of a Body in the same Distance describing a Circle, in dimidiate Ratio of the *Latus Rectum*, to that duplicate Distance. Seeing $AR = \frac{SA \times SA}{SP}$, it will be $C^2 : c^2 ::$

$\frac{SP \times SA \times S\dot{A}}{S\dot{P}} : SA^2 :: \frac{SP \times S\dot{A}}{S\dot{P}} : SA :: SP \times S\dot{A} :$ $SA \times S\dot{P}$; therefore from the given Relation SP to SA, the Ratio of C to c will be given.

EXAMPLE.

If the Force be reciprocally as the Power m of the Distance, that is, let $\frac{S\dot{P}}{SP^3 \times S\dot{A}} = \frac{b}{a^2 SA^m}$, and it will be $S\dot{P} : \frac{bSP^3 \times S\dot{A}}{a^2 SA^m}$; wherefore it will be $C^2 : c^2 :: SP \times S\dot{A} : \frac{bSP^3 \times SA \times S\dot{A}}{a^2 SA^m} :: a^2 SA^{m-1} : bSP^2$; whence, if we put $SP^2 = \frac{d^2 SA^{m-1}}{b} = \frac{m-1}{2} \frac{a^2 SA^{m-1}}{b}$, it will be $C^2 : c^2 :: a^2 SA^{m-1} : \frac{m-1}{2} a^2 SA^{m-1} :: m-1 : 2$; and moreover $C : c : \sqrt{2} : \sqrt{m-1}$.

But if we put $SP^2 = \frac{d^2 SA^{m-1}}{b - eSA^{m-1}} = \frac{m-1}{2} \frac{a^2 SA^{m-1}}{b - eSA^{m-1}}$ it will be C^2 to c^2, as $a^2 SA^{m-1}$ to $\frac{\frac{m-1}{2} a^2 bSA^{m-1}}{b - eSA^{m-1}}$ that is, as $b - eSA^{m-1}$ to $\frac{m-1}{2} b$; but the Ratio is $b - eSA^{m-1}$ to $\frac{m-1}{2} \times b$, less than the Ratio b to $\frac{m-1}{2} b$, or the Ratio 2 to $m-1$; whence will C to c be in a less Ratio, as $\sqrt{2}$ to $\sqrt{m-1}$.

Likewise

Likewise, if SP be taken $= \frac{d^2 SA^{m-1}}{b + e SA^{m-1}}$, C will be found to c in a greater Ratio than $\sqrt{2}$ to $\sqrt{m-1}$.

Corollary.

If a Body be moved in a Parabola, and the Centripetal Force tend to the Focus S, the Velocity of a Body will be to the Velocity of a Body describing a Circle in the same Distance every where, as $\sqrt{2}$ to 1. For in this Case $m = 2$, and $m - 1 = 1$. The Velocity of a Body in an Ellipsis is to the Velocity of a Body moving in a Circle to the same Distance, in a lesser Ratio than $\sqrt{2}$ to 1. And the Velocity in an Hyperbola is to the Velocity in a Circle, in a greater Ratio than $\sqrt{2}$ to 1.

If a Body be carried in a Nautical Spiral, its Velocity is every where equal to the Velocity of a Body, describing a Circle in the same Distance, for $m = 3$, and $m - 1 = 2$.

Problem.

Granted, that the Centripetal Force (whose absolute Quantity is known) be reciprocally, as the Square of the Distance; and let a Body be projected according to a given right Line, with a given Velocity, to find the Curve in which the Body is moved. (*See Fig.* 31.) *Fig.* 31.

Let a Body be projected according to a given right Line AB, with the given Velocity C. And seeing the absolute Quantity of the Centripetal Force is known; from thence will be given the Velocity, in which a Body might describe a Circle to the Distance SA, the same Force urging it; for it is equal to that which is sought, whilst a Body, with the same applicate, urging Force, uniformly falls through $\frac{1}{2}$ SA. Let that Velocity be c. From A in AB, erect the Perpendicular AK, and in it take AR, a fourth Proportional to c^2 C^2, and $\frac{SA^2}{SP}$, and

and AR will be the Radius of the Curvature in A. From R on AS let fall the Perpendicular RH, and from H on AR, the Perpendicular HK, and having drawn the right Line SK, the Axis will give the Position. Make the Angle FAK = Angle SAK; and if FA be parallel to SK, the Figure in which a Body is moved will be a Parabola. And if SK meet the Axis in F, and the Points S and F fall on the same Part of the Point K, the Figure will be an Hyperbola; but if the Points S and K fall on contrary Parts, the Figure will be an Ellipsis, whence the Section, in which a Body is moved, will be described by the Foci S and F, and the Axis = SA $\pm$ FA.

A Solution

A Solution *of the* Inverse Problem *of* Centripetal Forces.

LET VIL be any Curve, which a Body urging by Centripetal Force (See *Fig.* 32.) describes, tending to the Centre C: This Curve, the Right Lines IP, *Fig.* 32.
K*p*, cut in two Poi
which from the Ce
C*f*; also with the
draw CI.

The Centripetal

Theorem, altho' w
another Demonstrati
K*n* parallel to CI.
angles ICP, IKN,
Triangles IK*m*, and

I*p* or IP : IK :
PC : IP :
IN : IK :

$PC \times IN : IK^2 :$

Moreover, the Tim
is, as the Area, or the Triangle ICK, or its Duple $PC \times IK$; therefore, if the Time be given, $PC \times IK$ will be a constant Quantity. And having given the Time, the Centripetal Force is, as the *Lineola* K*n*, which is described by the urging Force; therefore the Centripetal Force is, as the *Lineola* K*n* drawn into the constant Quantity $\frac{1}{PC \times IK^2}$, that is, the Centripetal

Force

Force will be as $\frac{1}{PC^2 \times IK^2} \times \frac{Pp \times IK^2}{PC \times IN}$, or, as the Quantity $\frac{Pp}{PC^3 \times IN}$. Q. E. D.

The Velocity of a Body in any Place is as the Force run over directly in any least Time, and, as that Time *inversely*; and therefore, as $IK \times \frac{1}{PC \times IK}$, that is, the Velocity will be reciprocally, as the Perpendicular from the Centre on the Tangent.

If the Distance of a Body from the Centre be called x, and the Perpendicular on the Tangent be called p, IN will be $= \dot{x}$, and $Pp = \dot{p}$, and the Centripetal Force may be exhibited by $\frac{f^4 \dot{p}}{p^3 \dot{x}}$, by taking any Quantity for f^4.

Wherefore let us call the Ceutripetal Force φ, then will $\frac{f^4 \dot{p}}{p^3 \dot{x}} = \varphi$, and $\frac{f^4 \dot{p}}{p^3} = \dot{x} \varphi$, and by taking their Fluents $\frac{f^4}{2p^2} =$ Fluent of $\dot{x}\varphi$.

And when the Velocity of a Body is reciprocally, as the Perpendicular p, its Square may be exhibited by $\frac{f^4}{2p^2}$. Therefore, if the Velocity be called v, then will $v^2 = \frac{f^4}{2p^2} =$ Fluent of $\dot{x}\varphi$. But if A be the Place, from which the Body is to fall, that it may acquire the Velocity v in D or I, and from the Place of the Body D be erected the Perpendicular $DF = \varphi$; then will the Rectangle $DE \times DF = \dot{x}\varphi$. Now let BFG be a Curve Line, whose Ordinates exhibit the Centripetal Forces, or the Quantities φ. The Fluent of $\dot{x}\varphi$ will be the Curvilinear

vilinear Area ABFD $= v^2 = \frac{f^4}{2p^2}$: But if the Velocity, be that which is acquired by falling from an infinite Distance, v^2, the Fluent of $\dot{x}\varphi$ will be equal to the Area ODFO indefinitely protensed.

Hence will p be always given in finite Terms, when the Curvilinear Area can be expressed in finite Terms.

EXAMPLE I.

Let the Centripetal Force be reciprocally as the Power (m) of the Distance, that is, let $\dot{x}\varphi = \frac{g\dot{x}}{x^m}$. If the Velocity of a Body be that which is acquired, by falling from an infinite Distance; then will $v^2 = \frac{g}{\overline{m-1} \times x^{m-1}}$ $= \frac{f^4}{2p^2}$; and, in all these Cases, the Area indefinitely protensed is the finite Quantity. And a Body may be revolved in a Trajectory Velocity, whose Square may be made either greater or less than $\frac{g}{\overline{m-1} \times x^{m-1}}$; or equal to it. Therefore will $v^2 = \frac{f^4}{2p^2} = \frac{g}{\overline{m-1}\, x^{m-1}}$ $\pm e^2$.

Hence by these urging Forces three kinds of Curve, may be described according as e^2 is a positive Quantity or Negative, or none at all.

EXAMPLE 2.

If the Velocity be greater than that which is acquired by falling from an infinite Distance, $\frac{f^4}{2p^2}$ is made $= \frac{g}{\overline{m-1}\, x^{m-1}} + e^2$; but if the Velocity be

less,

lefs it will be $\frac{f^4}{2p^2} = \frac{g}{\overline{m-1}\,x^{m-1}} - e^2$; if equal, it will be $\frac{f^4}{2p^2} = \frac{g}{\overline{m-1}\,x^{m-1}}$.

Let $\frac{1}{2} f^4$ be $= a^2e^2$, and $\frac{1}{m-1} \times g = b^2e^2$. And if the Velocity of a Body be that which is acquired by falling from an infinite Diftance, $p^2 = \frac{a^2x^{m-1}}{b^2}$, or $p = \frac{ax^{\frac{m-1}{2}}}{b}$.

But if the Velocity be greater or lefs than this Velocity, it will be made as has been fhewn $\frac{f^4}{2p^2} = \frac{g}{\overline{m-1}\,x^{m-1}} \pm e^2 = \frac{\frac{1}{m-1} g \pm e^2x^{m-1}}{x^{m-1}}$. Whence for $\frac{1}{2} f^4$, and $\frac{g}{m-1}$, by putting their Values a^2e^2, and b^2e^2, it will be $\frac{a^2e^2}{p^2} = \frac{b^2e^2 \pm e^2x^{m-1}}{x^{m-1}}$, or $\frac{a^2}{p^2} = \frac{b^2 \pm x^{m-1}}{x^{m-1}}$, then will $p^2 = \frac{a^2x^{m-1}}{b^2 \pm x^{m-1}}$. Confequently, if the Centripetal Force be reciprocally, as the Cube of the Diftance, that is, if $m = 3$, and $m-1 = 2$, p^2 will be $= \frac{a^2x^2}{b^2}$, or $p^2 = \frac{a^2x^2}{b^2 + x^2}$, or finally $p^2 = \frac{a^2x^2}{b^2 - x^2}$.

In the firft Cafe it appears, that the Curve is a Logarithmical Spiral, for $p = \frac{ax}{b}$, for $b : a :: x : p$.

Therefore

Therefore by Reason of the constant Ratio of b to d, the Angle CIP will be every where constant.

Let us put $p^2 = \frac{a^2x^2}{b^2+x^2}$, and from this Supposition there arises three diverse Kinds of Curves, as a, is greater, less, or equal to b.

And First, let a be greater than b. (See *Fig.* 33.) *Fig.* 33.
With the Centre C, and to any given Distance, describe
a Circl
duced,
IP², P
x^2 —
$x^2 - a^2$
IN : I
a is gr
tity.
Let th
CY :
YX =

Let $x = \frac{c}{z}$, whence $\dot{x} = -\frac{c^2\dot{z}}{z^2}$, and $\frac{\dot{x}}{x} = -\frac{\dot{z}}{z}$. Also $x^2 - c^2 = \frac{c^4}{z^2} - c^2 = \frac{c^4 - c^2z^2}{z^2} = \frac{c^2}{z^2} \times \overline{c^2 - z^2}$: Whence $\sqrt{x^2 - c^2} = \frac{c}{z} \times \sqrt{c^2 - z^2}$:

Which Values being substituted, it will be $\frac{ba\dot{x}}{x\sqrt{x^2 - c^2}} =$

$\frac{-ha\dot{z}}{c\sqrt{c^2-z^2}}$. Let $a:e::n:1$. that is, let $a=ne$;

then will XY, or $\dot{y}=-\frac{nh\dot{z}}{\sqrt{c^2-z^2}}$. But $\frac{nh\dot{z}}{\sqrt{c^2-z^2}}$

is to $\frac{c\dot{z}}{\sqrt{c^2-z^2}}$, as nh to c, that is, in a given Ratio. Consequently, therefore their Fluents, if they begin together, will be in the same Ratio, that is HY, or y

will be to the Fluent of $\frac{C\dot{z}}{\sqrt{c^2-z^2}}$, as nh to c.

But if with the Centre C, and Radius CV $=c$, a Circle VL be described, and CG be $=z$, and $no=\dot{z}$,

the Arch mn will be $=\frac{C\dot{z}}{\sqrt{c^2-z^2}}\doteq$ to the Fluxion of the Arch Qm, when the Fluxion is a positive Quantity; but when it is negative, its Fluent is the Arc Vm, Compliment of the former. For the Compliment of the same Arc, hath the same Quantity denoting the Fluxion, only affected with different Signs; because, whilst one increaseth, the other decreaseth.

Hence is HY to Vm, as nh to c: but CV is to CH, as Ve : HY, that is, $c:h::$ Ve : $\frac{h\times Ve}{c}=$ HY;

wherefore it will be $\frac{h\times Ve}{c}$: Vm :: $nh:c$; whence Ve : Vm :: $n:1$.

Moreover, from the Nature of the Circle, it will be CG : CV :: CV : CT, when mT touches the Circle, that is, it will be $z:c::c:\frac{c^2}{z}=$ CT $=x$; hence if the Angle VCe be taken to the Angle VCm, as n to 1; and Ce be produced to K, as CK may be $=$ Secant CT, the Point K will be in the Curve sought.

Here

Here it is to be noted, if n be a Number, that is, if a be to c, or a to $\sqrt{a^2-b^2}$, as Number to Number, VI will become an Algebraic Curve; for in this Case, the Relation mG to the Sine of the Angle VCe is defined by an Equation, and thence will be had the Relation of the Sine of the Angle VCe to CT, or CK, by a determinate Equation; and thence will be given an Equation, which will express the Relation between the Ordinate intercepted beginning from the Point C. The Orders and Degrees of these Curves by an Algebraic Scale of Equations are different for the Magnitude of the Number n. In all these Curves so described, the Position of the Asymptote is determined by this Ratio. Let the Angle VCL be made to a right Angle, as n to 1. In that Angle the Distance of a Body from the Centre appears infinite. Now the Square of the Perpendicular on the Tangent PC $=\frac{a^2x^2}{b^2+x^2}$; where x it infinite, PC^2 $=\frac{a^2x^2}{x^2}$, or PC$=a$, draw the Perpendicular CR to CL, and equal to a right Line a, and if thro' R be drawn RS, parallel to the Right Line CL, this will be a Tangent to an infinite Distance, or the Asymptote to the Curve.

If a Body in any of these Curves, by descending come to the lower Apside; hence again it will ascend *in infinitum*, and will describe another Curve, like to the former, or by ascending, will describe a Portion like the same Curve..

These Curves may be described about the Centre by many Revolutions, before they begin to converge to the Asymptote, and the Angular Motion of the Right Line CK will be equal to as many Right Lines, as there are Unities in n.

EXAMPLE

Let $n=100$, twenty-five whole Revolutions will be made before the Distance from the Centre appears infinite.

 Having

Having augmented n, a remaining the same, c is diminished: for $\frac{a}{n} = c$, and $\frac{a^2}{n^2} = c^2 = a^2 - b^2$; whence $\overline{n^2 - 1} \times a^2 = n^2 b^2$; and therefore it will be $a^2 : b^2 :: n^2 : n^2 - 1$; wherefore, if b^2 come to the Equality of a^2, $n^2 - 1$ will also come to the Ratio of the Equality with n^2; and therefore n will be augmented, and in the same Ratio will c be diminished. Wherefore, let us put b^2 to be almost equal to a^2; therefore, as when the Difference is infinitely small, n becomes an infinite great Number, and the Radius of the Circle c will become infinitely small, or the Circle wlll be drawn into its Centre. But c vanishing thus, CT does not vanish at the same Time, if the Angle VCM is almost a right one: For in every Circle, tho' very small, the Secant of a Right Angle is an infinite Quantity. Wherefore this Curve, by Reason n being infinite, will go round the Centre in infinite Revolutions, before it will begin to converge to the Assymptote.

And when c vanishes, $b = a$, and $p = \frac{ax}{\sqrt{x^2 + a^2}}$. And because in every Case $\dot{y} = \frac{ba\dot{x}}{x\sqrt{x^2 + c^2}}$, when c vanishes (See *Fig.* 34.

aking the Fluents $y =$

ntity.

l Spiral, which hath
y Radius be drawn to
nd the Periphery of the
be raised the Perpendi-
ve in I, and the Right
a constant Right Line,
which Property it re-
the Subtangent of the

For

For let the Radius of a Circle $CE = b$, and Arc $VE = a$, let CI be called x, and YV be y. Because $ba = x \times y$ will be $\frac{ba}{x} = y$, and $\frac{bax}{x^2} = \dot{y}$. Moreover CY

: CI :: YX :

therefore is

that is $\dot{x}$:

If with th
scribed the A
tween the Ri
be equal to th
ing VL × CF
: CF :: VL :
If to CG fro
or FG, or a
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Asymptote to

Now let b be greater than a; and likewise (as in the former Case) will be found $KN = \frac{ax}{\sqrt{x^2 + b^2 - a^2}}$, and seeing b exceeds a, it will be $c^2 = b^2 - a^2$, a positive Quantity, and $KN = \frac{ax}{\sqrt{x^2 + c^2}}$, and by putting the Radius of the Circle $HY = b$, will be found $XY =$

bax

$\frac{ha\dot{x}}{x\sqrt{x^2+c^2}}$. Let us put $x = \frac{c^2}{z}$, and will $\dot{x} = -\frac{c^2\dot{z}}{z^2}$, and $\frac{\dot{x}}{x} = -\frac{\dot{z}}{z}$; also $x^2 = \frac{c^4}{z^2}$, and $x^2 + c^2 = \frac{c^4}{z^2} + c^2 = \frac{c^4+c^2z^2}{z^2} = \frac{c^2}{z^2} \times \overline{c^2+z^2}$; whence $\sqrt{x^2+c^2} = \frac{c}{z} \times \sqrt{c^2+z^2}$. These Values being substituted, it becomes $\frac{ha\dot{x}}{x\sqrt{x^2+c^2}} = -\frac{ha\dot{z}}{c \times \overline{c^2+z^2}|^{\frac{1}{2}}} = -\dot{y}$. For the Beginning of the Arc HY may be [redacted] $\frac{-ha\dot{z}}{\overline{c^2+z^2}|^{\frac{1}{2}}}$, increaseth

$\frac{nh\dot{z}}{\overline{c^2+z^2}|^{\frac{1}{2}}} = \dot{y}$, and

$h^2 : c^2$, that is, in

the Sector CXY

en Ratio. Where-

will be in the same

to begin. And the

Fluent of the Sector CXY is the Sector CVY, and the Fluent of $\frac{\frac{1}{2}c^2\dot{z}}{\overline{c^2+z^2}|^{\frac{1}{2}}}$ is the Sector of the Hyperbola, which is thus demonstrated.

With the Centre C, and Semiaxis Transversus CV
Fig. 35. $= c$ (See *Fig.* 35.) describe an equilateral Hyperbola, and from the two Points D and F, be drawn the Ordinates DB, EF, to the Conjugate Axis; likewise

likewise draw CD, CF. And the Increment or Fluxion of the Triangle BCD will be equal to BE × BD — the Sector DCF: Whence the Sector DCF (which is the Fluxion of the Sector CVD) will be equal to BE × BD — the Increment of the Triangle BCD. And if BC be called z, (by Reason of the Hyperbola) $BD^2 = BC^2 + CV^2 = z^2 + c^2$; whence $BD = \overline{c^2+z^2}|^{\frac{1}{2}}$, and $BE \times BD = \dot{z} \times \overline{c^2+z^2}|^{\frac{1}{2}}$. But the Triangle BCD is $\frac{1}{2}z \times \overline{c^2+z^2}|^{\frac{1}{2}}$; whose Fluxion is $\frac{1}{2}\dot{z} \times \sqrt{c^2+z^2} + \dfrac{\frac{1}{2}\dot{z} \times z^2}{\overline{c^2+z^2}|^{\frac{1}{2}}}$. This subtracted from $\dot{z} \times \overline{c^2+z^2}|^{\frac{1}{2}}$, and there will remain the least Sector CDF of the Hyperbola $= \frac{1}{2}\dot{z} \times \overline{c^2+z^2}|^{\frac{1}{2}} - \dfrac{\frac{1}{2}\dot{z} \times z^2}{\overline{c^2+z^2}|^{\frac{1}{2}}} = \dfrac{\frac{1}{2}\dot{z} \times c^2 + z^2 - \frac{1}{2}\dot{z} \times z^2}{\overline{c^2+z^2}|^{\frac{1}{2}}}$ $= \dfrac{\frac{1}{2}c^2\dot{z}}{\overline{c^2+z^2}|^{\frac{1}{2}}}$. Wherefore the Fluent of the Sector CDF is equal to the Fluent of $\dfrac{\frac{1}{2}c^2\dot{z}}{\overline{c^2+z^2}|^{\frac{1}{2}}}$; wherefore the Sector CVD will be the Fluent of $\dfrac{\frac{1}{2}c^2\dot{z}}{\overline{c^2+z^2}|^{\frac{1}{2}}}$. Moreover DT is a Tangent to the Hyperbola, and meets the Conjugate Axis in T. And from the Nature of the Hyperbola it is BC : CV :: CV : CT; that is $z : c :: c : \dfrac{c^2}{z} = CT = x$; and from hence arises the following

CONSTRUCTION.

With the Centre C, and Semiaxis Transversus CV (See *Fig.* 36.) describe an equilateral Hyperbola Vm; *Fig.* 36. and also a Circle Ve. Let the circular Sector CVe, be taken to the Hyperbolical CVm, as n to 1. Let the Line

Line Tm touch the Hyperbola in m, meeting the conjugate Axis in T; produce Ce to k, that Ck may be $=$ CT; and the Point k will be in the Curve sought; to wit, that Curve is ſuch, that if Ck be called x; the Perpendicular from C let fall on the Tangent will always be equal to $\frac{a\dot{x}}{\overline{b^2+x^2}|^{\frac{1}{2}}}$. When x is infinite, b^2 vaniſhes, and the Perpendicular is made $= a$; and then CR coincides with CV. If therefore on the conjugate Axis, CR be taken $= a$, and RS be drawn parallel to CV, this will be Aſymptote to the Curve.

If a be augmented, that $b^2 - a^2$ become infinitely ſmall, then c^2 will vaniſh, and $\frac{ba\dot{x}}{x \times \overline{x^2+c^2}|^{\frac{1}{2}}}$ becomes $\frac{ba\dot{x}}{x^2} = \dot{y}$; whence if the Fluents of theſe Quantities be taken, we ſhall have $\frac{ba}{x} = y =$ and $ba = xy$, that is, the Rectangle under the circular Arch, and the Diſtance of the Curve from the Centre will always be a given Quantity; and by this Reaſon, the Curve will be an Hyperbolical Spiral. Wherefore that Hyperbolical Spiral may be conceived to be formed, either by the Sector of the Circle, or Ellipſis, or by the Sector of the Hyperbola, whoſe *Axis Tranſverſus* is diminiſhed *in infinitum*, and in the ſame Ratio is n augmented.

Hence come we to that Caſe, where a leſs Velocity of a Body is that which is acquired by falling from an infinite Diſtance, and where $p^2 = \frac{a^2x^2}{b^2 - x^2}$; and here by the ſame Method of Reaſoning, as in the former Caſe, will be found KN $= \frac{a\dot{x}}{\overline{b^2 - a^2 - x^2}|^{\frac{1}{2}}}$, where it is neceſſary that b^2 be greater than a^2. Hence, if $b^2 - a^2$

be

be called c^2; then KN $= \frac{a\dot{x}}{\overline{c^2-x^2}|^{\frac{1}{2}}}$; and consequently XY, or $\dot{y} = \frac{ba\dot{x}}{x\times\overline{c^2-x^2}|^{\frac{1}{2}}}$.

Let $x = \frac{c^2}{z}$, and $\frac{\dot{x}}{x}$ will be $= -\frac{\dot{z}}{z}$, or $\frac{ba\dot{x}}{x} = -\frac{ba\dot{z}}{z}$, and c^2-x^2 will be $= \frac{c^2}{z^2}\times z^2-c^2$, which Values being substituted is made $\frac{-ba\dot{z}}{c\times\overline{z^2-c^2}|^{\frac{1}{2}}} = \frac{ba\dot{x}}{x\times x^2-}$

ning of the Arch VX is

with the Fluent of $\frac{-}{c\times}$

$= \frac{1}{2}b\dot{y} =$ Sector CXY

But $\frac{\frac{1}{2}nb^2\dot{z}}{\overline{z^2-c^2}|^{\frac{1}{2}}}$ is to $\frac{}{z^2}$ in a constant Ratio. W

same Ratio, that is, t

$z^2-c^2|^{\frac{1}{2}}$

will be to the Fluent of $\frac{\frac{1}{2}c^2\dot{z}}{\overline{z^2-c^2}|^{\frac{1}{2}}}$, as nb^2 to c^2. And the Fluent of $\frac{1}{2}b\dot{y} =$ Sector CVX; and the Fluent of $\frac{\frac{1}{2}c^2\dot{z}}{\overline{z^2-c^2}|^{\frac{1}{2}}}$ is the Sector of the Hyperbola, which is thus demonstrated.

DEMONSTRATION.

With the Centre C, and *Semiaxis Transversus* CV$=c$, describe an equilateral Hyperbola; and from two Points infinitely near to B and D, let the two right Lines BE, DF be drawn as Ordinates to the Axis; also draw CB, CD, and the Fluxion or Increment of the Triangle CBE

Fig. 37. = Triangle CBD + BE × EF (See *Fig.* 37.) whence the Triangle CBD, or the small Sector CBD will be ment of the Triangle CBE—BE × EF. BE will be $=\overline{z^2-c^2}|^{\frac{1}{2}}$, and BE × EF . Also the Triangle CBE $=\frac{1}{2}z\times$

Fluxion is $\frac{1}{2}\dot{z}\times\sqrt{z^2-c^2}+\frac{\frac{1}{2}\dot{z}\times z^2}{\overline{z^2-c^2}|^{\frac{1}{2}}}$;

subtract $\dot{z}\times\overline{z^2-c^2}|^{\frac{1}{2}}$, the small Sector

$-\frac{1}{2}\dot{z}\times\overline{z^2-c^2}|^{\frac{1}{2}}=\frac{\frac{1}{2}\dot{z}\times z^2-\frac{1}{2}\dot{z}\times\overline{z^2-c^2}}{\overline{z^2-c^2}|^{\frac{1}{2}}}$

Whence it appears, that the Sector CBE

$\frac{\frac{1}{2}c^2\dot{z}}{\overline{z^2-c^2}|^{\frac{1}{2}}}$. Moreover, if BT, the Hyperbola, meet the Transverse Axis Nature of the Hyperbola it is CE : CV that is

$$z:c::c:\frac{c^2}{z}=\text{CT}=x.$$

Hence we have deduced the following

CONSTRUCTION.

With the Centre C, and *Semi-Transversus* CV $=c$, describe an Equilateral Hyperbola VB, and a Circle C*e*G from

from the Centre C (See *Fig.* 38.) draw the right Line CB to the Hyperbola, and let the Tangent of the Hyperbola, BT, meet the *Axis Transversus* in T. Let the Sector of the Circle CVe, which is to the Hyperbolical Sector, CVB, be as n to 1. In Ce let CK=CT, and K will be the Point in the Curve sought, whose Perpendicular from the Centre C, let fall on the Tangent K, if CK be called x, is equal to $\frac{ax}{\overline{b^2 - x^2}|^{\frac{1}{2}}}$. *Fig.* 38.

And in this Curve, by the urging Centripetal Force, which is reciprocally as the Cube of the Distance, the Body will be moved, if according to the Direction of the Tangent it go off with a just Velocity.

When the Velocity, with which a Body in any trajectory is moved, be reciprocally as p, by assuming any constant Quantity a, it may be exhibited by $\frac{a}{p}$; and if right Lines be drawn as Ordinates to the Axis CV, which are reciprocally as the Cubes of their Distances from the Centre, or, as the Centripetal Forces; and by this Ratio are formed the Curvilinear Figure, its Area indefinitely extended may always be exhibited by $\frac{b^2}{x^2}$, as appears from the Quadratures. But that Area is as the Square of the Velocity, which is acquired, by falling from an infinite Distance, and the Velocity sought in this Case will be as $\frac{b}{x}$. Hence, if that Velocity be called y, and the Velocity with which a Body is moved in a Trajectory be called v, and a and b be taken such, as in any one Distance from the Centre $y : v :: \frac{b}{x} : \frac{a}{p}$, every where it will be in all Distances $y : v :: \frac{b}{x} : \frac{a}{p} :: p : \frac{ax}{b}$; whence, if $y = v$, p will be

 $= \frac{ax}{b}$

$= \frac{ax}{b}$; and the Curve will be a Nautical Spiral described by this Velocity; or Circle, p being $= x$, and $a = b$.

If y be greater than v, then p will be greater than $\frac{ax}{b}$, and it will be (as appear from the preceding) $= \frac{ax}{\overline{b^2 - x^2}|^{\frac{1}{2}}}$. And the Curve will be constructed by the Hyperbolical Sector, as was shewn in the last Case, where the Distance of a Body from the Centre *per Concursum* of the Tangent of the Hyperbola, with the Transverse Axis is determined. If y be less than v, but in so small a Ratio that b may be greater than a, the Curve will be formed by the same Hyperbolical Sector: But the Distance of a Body from the Centre is taken from the Concourse of the Tangent with the conjugate Axis.

If $y : v :: p : x$, it will then be $a = b$, and the Curve is an Hyperbolical Spiral, where $p = \frac{ax}{\overline{a^2 + x^2}|^{\frac{1}{2}}}$. Hence if from any Place be projected a Body, according to a given right Line with that Velocity, which is to the Velocity sought by falling infinitely, as the Distance of a Body from the Centre to the Perpendicular from the Centre to the Line of Direction let fall, that Body will be moved in an Hyperbolical Spiral.

Lastly, if v be so much the greater than y; as likewise a greater than b, the Curve will be constructed by Circular Sectors, and by this Ratio having the Velocity given, the Relation of a and b might always be determined, as the Curve will be described, in which, a Body will be moved with that Velocity; and again having the Curve given, or a, and b, the Velocity will be found wherein the Curve is described.

The Areas of all Curves (except the Circle) which by this urging Centripetal Force may be described, are perfect quadrable. For First in the Logarithmical Spiral; because $p = \frac{ax}{b}$, KN will be $= \frac{ax}{\overline{b^2 - a^2}|^{\frac{1}{2}}} = \frac{ax}{c}$

by

by putting $b^2 - a^2 = c^2$ (See *Fig.* 32.) therefore will *Fig.* 32.
the Triangle CKI $= \frac{\frac{1}{2}a x \dot{x}}{c}$, whose Fluent is $\frac{a x^2}{4c}$ = Area of the Curve.

If p be $= \frac{ax}{\overline{b^2 + x^2}|^{\frac{1}{2}}}$, and a greater than b, it has been shewn, that KN $= \frac{a\dot{x}}{\overline{x^2 - c^2}|^{\frac{1}{2}}}$; w[illegible] KN ...

$\frac{1}{2}$ CI $= \frac{\frac{1}{2}a x \dot{x}}{\overline{x^2 - c^2}|^{\frac{1}{2}}}$, whose Fluent is $\frac{1}{2}a$ [illegible]
Area of the Curve.

But if a be less than b, KN $=$ [illegible] x^2 [illegible]

$\times \frac{1}{2}$CI $= \frac{\frac{1}{2}a x \dot{x}}{\overline{x^2 + c^2}|^{\frac{1}{2}}}$, its Fluent $= \frac{1}{2}a \times$ [illegible]
= Area of the Curve. Put $x = o$, and
— Q $= o$. Whence Q $= \frac{1}{2}ac$, and th[illegible]
Curve is made $= \frac{1}{2}a \times \overline{x^2 + c^2}|^{\frac{1}{2}} - \frac{1}{2}ac$.

In the Hyperbolical Spiral, c vanishes, [illegible]
of the Curve is $\frac{1}{2}ax$.

If $p = \frac{ax}{\overline{b^2 - x^2}|^{\frac{1}{2}}}$, it has been shewn [illegible]

$\frac{a\dot{x}}{\overline{c^2 - x^2}|^{\frac{1}{2}}}$; whence $\frac{1}{2}$ CI $\times$ KN $=$ [illegible] c^2 [illegible]
Fluent is Q$-\frac{1}{2}a \times \overline{c^2 - x^2}|^{\frac{1}{2}}$ = Area. Make $x = o$, and will Q$-\frac{1}{2}ac = o$, or Q $= \frac{1}{2}ac$. Whence the Area of the Curve will be always equal to $\frac{1}{2}ac - \frac{1}{2}a \times \overline{c^2 - x^2}|^{\frac{1}{2}}$. Make $c^2 - x^2 = o$, or $c = x$, and the Area of the Curve is $\frac{1}{2}ac$.

Whence, if the Beginning of the Area is not taken from the Beginning of x, or where $x = o$; but when $x = c$, it is a *Maximum*, that is, if Area begin from V (See

Fig. 38. (See *Fig.* 38.) the Area will always be equal to $\frac{1}{2} a \sqrt{c^2 - x^2}$.

PROBLEM I.

Fig. 39. *Upon the right Line* AG, *as the Axis* (*See Fig.* 39.) *from the Point* A *to draw infinite Curves, such as* ABD *of that Nature, as the contact Radii, and every where drawn in all the Points* B, *may be cut by* BO *of the Axis* AG *in* C, *in a given Ratio*, viz. *let* BO *be to* BC, *as* 1 *to n.*

Then to construct the Trajectories EBF, *cutting the first Curves normally.*

Draw the Ordinate BH normally to the Axis AG, let the Abscissa AH $= z$, Ordinate HB $= x$. the Curve AB $= v$. Then by the direct Method of Fluxions BC will be $= \frac{\dot{v}}{\dot{z}} x$, and v flowing ~~uniformly~~, BO $= \frac{\dot{v}\dot{x}}{\ddot{z}}$. Whence by the Condition of the Problem BO $\left(\frac{\dot{v}\dot{x}}{\ddot{z}}\right)$: BC $\left(\frac{\dot{v}}{\dot{z}} x\right)$ :: 1 : n; consequently $\ddot{z} x - n\dot{z}\dot{x} = o$.

This Equation being compared with the second Form of Fluxions according to Dr. *Taylor*'s, at the End of Proposition 6. of the Method of Increments, is found $\dot{z} x^{-n} = \dot{v} a^{-n}$ a being the given Line, by whose Value ABD may be accommodated to any Problem.

For $\dot{v}$ writing its Value $\overline{\dot{x}^2 - \dot{z}^2}|^{\frac{1}{2}}$, the Equation becomes $\dot{z} x^{-n} = \dot{v} a^{-n}$ in this $\dot{z} = \frac{\dot{x} x^n}{\overline{a^{2n} - x^{2n}}|^{\frac{1}{2}}}$.

Whence x being given, z becomes known also by the Quadrature

Quadrature of the Curve, whose Abscissa being x, its Ordinate is $\frac{x^n}{\overline{a^{2n}-x^{2n}}|^{\frac{1}{2}}}$.

Let σ and τ be whole Numbers, either Affirmative or Negative, such as it may be the most simple of Curves coming out in this Manner, whose Abscissa is y, and Ordinate is $y^{\frac{1-n+2\sigma n}{2n}} \times \overline{a-y}|^{\tau-\frac{1}{2}}$; then that will be the most Simple of all Curves, by whose Quadrature is given the Abscissa z from the given Ordinate x.

The Curve ABD is Geometrical; as often as the Reciprocal of any unequal Number is taken for n.

In the preceding we have consider'd the Curve ABD, as Concave towards the Axis AG, where x, the greatest Ordinate, is equal to the given Line a, which let us call the Parameter of the Curve. And in this Case, the Curve meets the Axis, *Actu*. Whence if the Fluent of $\frac{\dot{x}\, x^n}{\overline{a^{2n}-x^{2n}}|^{\frac{1}{2}}}$ be duly taken, that is, that z and x may vanish together, the Curve will pass thro' the given Point A, as the Problem requires.

But if the Curve ABD be sought, which is convex towards the Axis, after the same Manner, we get this Equation $\dot{z} = \frac{a^n \dot{x}}{\overline{x^{2n}-a^{2n}}|^{\frac{1}{2}}}$, which likewise may be derived from the former Equation by changing the Sign of n. And in this Case, the Curve ABD is Geometrical, as often as the reciprocal of any even Number is taken for n. And in this Case, the Ordinate x being the least of all is equal to the Parameter a; and therefore the Curve no where meets the Axis. Wherefore the Problem is limited to the former Case.

From what has been said it is easy to gather, that all the Curves ABD are similar among themselves, and similarly

similarly placed about the given Point A, their Homologous Sides being proportional to the Parameters a.

Hence it is $\dot{v} : \dot{z} :: a^n : x^n$. But BC : BH :: $\dot{v} : \dot{z}$, whence it is BC : BH :: $a^n : x^n$. But from the Condition of the Problem BC is a Tangent to the Curve EBF sought. Wherefore, if now we take AH (z) and BH (x) for the Co-ordinates of the Curve EBF, the Curve EB being r; then by the direct Method of Fluxions $\dot{r} : -\dot{x} ::$ (BC : BH ::) $a^n : x^n$. Whence $\frac{x^n}{a^n} = \frac{-\dot{x}}{\dot{r}}$

In the Curve ABD suppose the Equation $\dot{z} = \frac{\dot{x} x^n}{\overline{a^{2n} - x^{2n}}|^{\frac{1}{2}}}$ to be transformed into this Equation not affected with Radical Signs $\dot{z} = A\dot{x}\frac{x^n}{a^n} + B\dot{x}\frac{x^{3n}}{a^{3n}} +$ &c. Then by taking their Fluents $z = \frac{1}{n+1} A \frac{x^{n+1}}{a^n} + \frac{1}{3n+1} B \frac{x^{3n+1}}{a^{3n}} +$ &c. by introducing no new Co-efficient, because by the Condition of the Problem z and x is to increase together. Hence, instead of $\frac{x^n}{a^2}$ by substituting its Value $\frac{-\dot{x}}{\dot{r}}$. Then will $z = \frac{1}{n+1} Ax\frac{-\dot{x}}{\dot{r}} + \frac{1}{3n+1} Bx\frac{-\dot{x}^3}{\dot{r}^3} +$ &c.

Now

Now let r flow uniformly, and a being a permanent Quantity, let $\frac{-\dot{x}}{\dot{r}} = \frac{s^n}{a^n}$. The Value of $\frac{-\dot{x}}{\dot{r}}$, being substituted in the Equation last found, and drawing the Equation into $\frac{s}{x}$, it becomes $\frac{zs}{x} = \frac{1}{n+1} A \frac{s^{n+1}}{a^n} + \frac{1}{3n+1} \times B \frac{s^{3n+1}}{a^{3n}} +$ &c. which in Fluxions is $\frac{\dot{s}zx + s\dot{z}x - sz\dot{x}}{x^2} = A\dot{s}\frac{s^n}{a^n} + B\dot{s}\frac{s^{3n}}{a^{3n}} = \frac{\dot{s}s^n}{\overline{a^{2n} - s^{2n}}|^{\frac{1}{2}}}$. Which last is manifest from the Analogy of the Series $A\dot{x}\frac{x^n}{a^n} +$ &c. $A\dot{s}\frac{s^n}{a^n} +$; hence for s and $\dot{s}$ having substituted their Values, being collated from the Equation $\frac{-\dot{x}}{\dot{r}} = \frac{s^n}{a^n}$, the Equat. is $n\dot{x}^2\dot{z}z - \ddot{x}x\dot{z}z - nx\dot{x}\dot{z}^2 - \ddot{x}\dot{x}x^2 = 0$, which is reduced into first Fluxions after the following Manner.

In the last Term $-\ddot{x}\dot{x}x^2$, instead of $\ddot{x}\dot{x}$, write its Value $-\ddot{z}\dot{z}$, then the Equation applied to $\dot{z}$ is made $n\dot{x}^2z - \ddot{x}xz - nx\dot{x}\dot{z} + xx\ddot{z} = 0$, which Equation drawn into x^{-n-1} is the Fluxion of $-\dot{x}x^{-n}z + x^{1-n}\dot{z} = a^{1-n}$

$a^{1-n}\dot{r}$; (a and $\dot{r}$ not being Fluents.) Therefore $-\dot{x}x^{-n}$ $z+x^{1-n}\dot{z} = a^{1-n}.\dot{r}$, or $\dot{z}x-\dot{z}x \times a^{n-1} = \dot{x}x^{n}$. The Fluxionary Equation of the first Step is to the Curve sought EBF.

But in this Equation, a is the Value of the Ordinate BH, when the Point H falls on the Point A.

It is not easy, that $\dot{z}x-\dot{z}x \times a^{n-1} = \dot{r}x^{n}$, n remaining in general Terms, to reduce it to an Equation involving only the Fluents, or *ad quadraturam Curvarum*. But the Points of the Curve EBF, may easily be found by the Description of the Curve ABD, and of any Geometrical Curve. Here by a Geometrical Curve, I mean, whose Equation is not in Fluxions, nor Fluents in the Indices of the Powers. For let the Curve ABD be cut in B, whose Parameter let be a, from the Geometrical Curve, whose Equation is $\dot{z}a^{n}x^{n} - za^{n}x^{n} = xa^{n} \times \overline{a^{2n}-x^{2n}}|^{\frac{1}{2}}$, and that Point of Intersection B will be to one of the Trajectories sought, to wit, which passeth through the Point E, AE being $= a$, and to the Normal AG. Consequently, if ABD be a Geometrical Curve, then EBF will be a Geometrical Curve also.

SCHOLIUM.

Also by any other Method may be found $\overline{\dot{z}x-\dot{z}x} \times a^{n-1} = \dot{r}x^{n}$. For by a certain Analysis, I made $\frac{\dot{a}}{a} = \frac{\dot{r}\dot{r}}{zz+xx}$. Which being compared with $\frac{x^{n}}{a^{n}} = \frac{\dot{z}-\dot{x}}{r}$, by rejecting a and $\dot{a}$, then we get $\overline{\dot{z}x-\dot{z}x} \times a^{n-1} = \dot{r}x^{n}$.

EXAMPLE.

EXAMPLE.

Let $n = 1$, in which Case ABD is a Semicircle described by the Diameter AG, and likewise EBF is a Semicircle described by the Diameter AE; and in this Case $\frac{\dot{x}x^{n}}{\overline{a^{2n}-x^{2n}}|^{\frac{1}{2}}} = \frac{\dot{x}x}{\overline{a^{2}-x^{2}}|^{\frac{1}{2}}}$, whence $\dot{z} = \frac{\dot{x}x}{\overline{a^{2}-x^{2}}|^{\frac{1}{2}}}$.

Consequently z

a Circle describ

Likewise, wri

$\dot{r}x^{n}$ is made $\dot{x}x$

$\dot{r}$ by the Help

$= -\dot{x}$; conseq

-x

$x+a$, which Equation is to a Circle described by the Diameter AE $= a$, which was to be done.

Of the Length of Curve Lines.

LEMMA.

TO divide the Sum of two Squares, into two other Squares.

Let $\dot{z}^2$, $\dot{s}^2$ be the two given Squares, whose Sum $\dot{z}^2+\dot{s}^2$ is to be divided into two other Squares $\dot{x}^2$, $\dot{y}^2$; and let m and n be any two Numbers taken at Pleasure.

Now from the Condition of the Problem $\dot{x}^2+\dot{y}^2 = \dot{z}^2+\dot{s}^2$, whence (it is manifest *ex Diaphonto*) $\dot{x}$ will be $= \frac{\overline{mm-nn}\times\dot{z}+2mn\dot{s}}{mm+nn}$, $\dot{y} = \frac{\overline{nn-mm}\times\dot{s}+2mn\dot{z}}{mm+nn}$.

Q. E. I.

PROBLEM I.

To find innumerable Curves, which may be of the same Length, with any proposed Curve, whether it be Algebraical or Transcendental.

Let z, s denote the Co-ordinates of the proposed Curve; and x, y, the Co-ordinates of the Curve sought, which let be of the same Length with the proposed; whence from the Elements of Curves $\dot{x}^2+\dot{y}^2 = \dot{z}^2+\dot{s}^2$. Therefore by the preceding *Lemma*,

$$\dot{x} = \frac{\overline{mm-nn}\times\dot{z}+2mn\dot{s}}{mm+nn}.$$

$$\dot{y} = \frac{\overline{nn-mm}.\times\dot{s}+2mn\dot{z}}{mm+nn}.$$

Whose

Whose Integrals are

$$x = \frac{\overline{m^2-n^2} \times z + 2mns}{mm+nn}.$$

$$y = \frac{\overline{n^2-m^2} \times s + 2mnz}{mm+nn}.$$

And thus becomes known the Co-ordinates, x, y, of one of the Curves sought; likewise from this One will be found a Second, from the Second, a Third; and so will innumerable Curves be found.

Q. E. I.

PROBLEM 2.

Suppose $\left.\begin{array}{l} x^{y} = 5000 \\ y^{x} = 3000 \end{array}\right\}$ Quere x and y?

SOLUTION.

Put $5000 = a$; $3000 = b$; it is evident x is something above 4, but under 5, and y something under 6, but more than 5. $c =$ Hyperbolical Logarithm of 5000 $= 8.517193$; $d =$ Hyperbolical Logarithm of 3000 $= 8.0061$. $m =$ Hyperbolical Logarithm of $4 = 1.3863$; $n =$ Hyperbolical Logarithm of $5 = 1.609436$; then by the Question,

$4+z=x$, $5+v=y$, from what has been said above; then (Log. $4+z=$) $m + \frac{z}{4} - \frac{z^2}{32} + \frac{z^3}{192} - \frac{z^4}{1024} + \frac{z^5}{5120}$, &c. $\times 5+v = c$.

(Log. $5+v=$) $n + \frac{v}{5} - \frac{v^2}{50} + \frac{v^3}{375} - \frac{v^4}{2500} + \frac{v^5}{15625}$, &c. $\times 4+z = d$. Hence $\frac{d}{4+z} - n = \frac{v}{5} - \frac{v^2}{50} + \frac{v^3}{375} - \frac{v^4}{2500} + \frac{v^5}{15625}$ &c.

Put

Put $g = d - 4n$, $f = \frac{1}{5}$; $h = \frac{1}{50}$; $k = \frac{1}{375}$, and then $\frac{g-nz}{4+z} = fv - hv^2 + kv^3 - lv^4 + pv^5$, &c. and by Reversion of Series we have $v = \frac{g-nz}{4f+fz} + \frac{hg^2 - 2hgnz + hn^2z^2}{4f^3+f^3z}$, &c. Consequently by the first Step, $1 + \frac{z}{4} - \frac{z^2}{32} + \frac{z^3}{192} - \frac{z^4}{1024}$, &c. $\times$ $\frac{20f^3+5f^3z+f^2g-f^2nz}{4f^3+f^3z} + \frac{hg^2-2hgnz+hn^2z^2}{4f^3+f^3z} = c$; and reducing all, and transposing the Terms, we have in Numbers, *viz* $.17402 - .0320z^2 + .0182z^3 + .0025z^4$, &c. $= .1043$. Now, if we revert the Series, the Value of z will be $= .7001$, nearly; hence $x = 4.7001$; and $y = 5.51$.

Q. E. I.

PROBLEM 3.

Suppose $x^{x^x} = 123456789$; quere x?

SOLUTION.

Let $4 - z = x$, then by the Nature of Exponentials, we shall have $l : \overline{4-z} \times \overline{16-8z+z^2} = l : 123456789$ $= 18.631400$; whence by Reversion of Series $z = .24655$. and consequently $x = 3.75345$.

Q. E. I.

PROBLEM 4.

Suppose $x^x + x^{\frac{1}{x}} = 100$. Quere x?

Put $n + z = x$, then *per* Question $\overline{n+z}^{n+z} + \overline{n+z}^{\frac{1}{n+z}} = 100 = b$. To find a Series that will express

press the Value of $\overline{n+z}|^{n+z}$ very near, put $\overline{n}|^{n} = d$, $m = l:n$, $c = m+1$, and let $d+v = \overline{n+z}|^{n+z}$; then by the Nature of Logarithms it will be

$$\overline{n+z} \times l:\overline{n+z} = mn + cz + \frac{z^2}{2n} - \frac{z^3}{6n^2}, \text{ \&c.}$$

$$= l:d + \frac{v}{d} - \frac{v^2}{2d^2} + \frac{v^3}{3d^3}, \text{ \&c.}$$ Here mn is $= l:d$.

$$\therefore \frac{v}{d} - \frac{v^2}{2d^2}, \text{ \&c.} = cz + \frac{z^2}{2n} - \frac{z^3}{6n^2}, \text{ \&c.}$$ and by reverting the Series $v = dcz + \frac{2nc^2d+d}{2n} z^2 + \frac{6n^2c^3d + 6ncd - d}{6n^2} z^3$, &c.

$$\therefore d+v = \overline{n+z}|^{n+z} = d + dcz + \frac{2nc^2d+d}{2n} z^2, \text{ \&c.}$$

Secondly, To find a Series that will express the Value of $\overline{n+z}|^{\frac{1}{n+z}}$, put $\overline{n}|^{\frac{1}{n}} = g$, $m = l:n$, $e = m-1$, and let $g+v = \overline{n+z}|^{\frac{1}{n+z}}$; then *per* Log. $\frac{1}{n+z} \times l:\overline{n+z} = l:g+v$; but $\frac{1}{n+z} = \frac{1}{n} - \frac{z}{n^2} + \frac{z^2}{n^3}$, &c. $\therefore \frac{1}{n+z} \times l:\overline{n+z} = \frac{m}{n} - \frac{ez}{n^2} + \frac{2m-3}{2n^3} z^2$, &c. $= l:g + \frac{v}{g} + \frac{v^2}{2g^2}$, &c. here $\frac{m}{n} = l:g$. $\therefore \frac{v}{g} - \frac{v^2}{2g^2}$, &c. $= -\frac{ez}{n^2} + \frac{2m-3}{2n^3} z^2$, &c. And by reverting the Series, as above, it will be

$$v = -\frac{gez}{n^2} + \frac{2mng - 3ng + e^2g}{2n^4} z^2, \text{ \&c.}$$

$$\therefore g+v = \overline{n+z}|^{\frac{1}{n+z}} = g - \frac{gez}{n^2} + \frac{2mng - 3ng + e^2g}{2n^4} z^2, \text{ \&c.}$$

Now

Now if for $\overline{n+z}^{n+z}$, and $\overline{n+z}^{\frac{1}{n+z}}$, their Values are substituted in the above Equation, it will be

$$d+g+\frac{dcn^2-ge}{n^2}z+\frac{2c^2n^4d+dn^3+2mng-3ng+e^2g}{2n^4}$$

$z^2+\frac{6n^2c^3d+6ncd-d}{6n^2}z^3$, &c. $=b$. Put $b-g-d=q$, and let $az+bz^2+cz^3$, &c. $=q$; then by reverting the Series, it will be $z=\frac{q}{a}-\frac{bq^2}{a^3}+\frac{2b^2-ac}{a^5}q^3$, &c. If n be assumed nearly $=x$, the above Series will converge very swift, if not, so very slow as not to be fit for Use. I find in the above Equation, that it is impossible for $x^{\frac{1}{x}}$ to be greater, or so great as 2. Wherefore I put $x^x=98$, by which x is nearly $=3.59=n$, then the above Series converges so very swift, that the two first Terms will give $z=.00098$ ∴ $x=3.59098$.

Q. E. I.

Problem 5.

Quere the Value of x, when $x^{\frac{1}{x}}$ is a *Maximum*.

It is evident from the Nature of the Problem, that the Fluxion of $x^{\frac{1}{x}}$ being $=0$. the Fluxion of its Logarithm, viz. $\frac{\dot{x}}{xx}-\frac{\dot{x}}{xx}\times l:x$ is also $=0$. Whence (dividing by $\frac{\dot{x}}{xx}$) $1-l;x=0$, and $l:x=1$, which in *Brigg*'s Form is .43429, &c. and the Number answering thereto is $2.71828=x$.

Q. E. I.

Problem

PROBLEM 6.

Given $x^x - x = y$ } Quere x and y?
and $y^x + y = x$ }

Put $1+z=x$, and $1-v=y$; then by Transposition $\overline{1+z}^{1+z} = 2+z-v$, and $\overline{1-v}^{1+z} = z+v$.

Now by Sir *Isaac Newton*'s Universal Theorem $\overline{P+PQ}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ$, &c.

Hence $\overline{1+z}^{1+z} = 1+z+z^2+\frac{z^3}{2}+\frac{z^4}{3}+\frac{z^5}{12}+\frac{3z^6}{40}$, &c. $= 2+z-v$. And $\overline{1-v}^{1+z} = 1-v-zv+\frac{v^2\times\overline{z+z^2}}{2}+\frac{v^3\times\overline{z-z^3}}{6}+\frac{v^4\times\overline{2z-z^2-2z^3+z^4}}{24}$, &c. $= z+v$. By the first Step, $v = 1-z^2-\frac{z^3}{2}-\frac{z^4}{3}-\frac{z^5}{12}-\frac{3z^6}{40}$, &c. and by the second Step, $1-2v-zv+\frac{v^2\times\overline{z+z^2}}{2}+\frac{v^3\times\overline{z-z^3}}{6}+\frac{v^4\times\overline{2z-z^2-2z^3+z^4}}{24}$, &c. $= z$.

Then if for v, and its several Powers in the fourth Step, we substitute its Value found in the third Step, and reduce the Co-efficients to one Denomination, we shall have

$$-1-\frac{1}{4}z+\frac{59z^2}{24}-\frac{z^3}{12}-\frac{13z^4}{24}+\frac{65z^5}{36}+\frac{27z^6}{15}, \text{ \&c.} = z.$$

And by Transposition and Substitution, (Sixth Step) $az-bz^2+cz^3+dz^4$, &c. $= 1$. And by the Method of of Reversion of Series $-\frac{1}{a}+\frac{b}{a^3}-\frac{2b^2+ac}{a^5}-\frac{5abc-a^2d+5b^3}{a^7}$, &c. $=z$. Hence $z=.748$ nearly; and consequently $x=1.748$, and $y=.9058$.

Q. E. I.

PROBLEM 7.

$$\text{Let } \overline{x)}^{x^x}+x^x-x^{-x}+x+x^{\frac{1}{x}}=200. \text{ Quere } x?$$

SOLUTION.

Let $x=a$, then by a Tryal or two, x is easily found to be more than 2, and less than 3. Therefore put $x=2+y$, and $m=$ Hyperbolical Logarithm thereof; then $2m+my=$ Hyperbolical Logarithm of x^x. Therefore by the Nature of Hyperbolical Logarithms $1+2m+my+\frac{2m^2+4m^2y+m^2y^2}{2}+\frac{2m^3+10m^3y+6m^3y^2+m^3y^3}{6}$, &c. $=x^x$.

Now m is $=\frac{1+y}{2+y}+\frac{1+2y+y^2}{8+8y+2y^2}+\frac{1+3y+3y^2+y^3}{24+36y+18y^2+3y^3}$ &c. And by writing this, instead of m, in the above Equation, and reducing the Fractions, &c. we have $5.4819+2.6498y+2.8333y^2+.55903y^3-.35245y^4$, &c. $=x^x$, which multiplied by m, gives the Hyperbolical Logarithm of $\overline{x)}^{x^x}=4.00186+4.105y+2.1916y^2+1.4837y^3-.339y^4-.1666y^5+.1056y^6-.058y^7$, &c. which put $=z$. Then $1+z+\frac{z^2}{2}+\frac{z^3}{2.3}+\frac{z^4}{2.3.4}$, &c. $=\overline{x)}^{x^x}$. But as this will not converge, let $4.00186=$ Hyperbolical Logarithm of 54.677 be deducted

ducted from z; and then we have $4.105y + 2.1916y^2$, &c. $= z - 4.00186$, which put $= n$. Then by the Nature of Hyperbolical Logarithms, $1 + n + \frac{n^2}{2} + \frac{n^3}{6} + \frac{n^4}{24}$, &c. $= \frac{\overline{x^x}|^x}{54.677}$, and restoring n, and writing the Value thereof in the Series, &c. we have $1 + 4.10966y + 10.2677y^2 + 19.8534y^3 + 33.0543y^4 + 45.173y^5 + 61.804y^6$, &c. $= \frac{\overline{x^x}|^x}{54.677}$, and by Multiplication, $54.677 + 224.704y + 561.347y^2 + 1085.052y^3 + 1807.311y^4 + 2470.092y^5 - 3379.27y^6 = \overline{x^x}|^x$.

Now x^x being already found as above, $x^{-x} = \frac{1}{x^x}$ is easily found $= .1824 - .08818y - .05166y^2 + .05199y^3 - .000785y^4 - .0265y^5 + .00577y^6$, &c. and by changing the Signs, becomes $-x^{-x}$.

Again, (m being $=$ the Hyperbolical Logarithm of x) $\frac{m}{x} =$ Logarithm of $x^{\frac{1}{x}}$, and (x being $= 2 + y$,) $1 + \frac{m}{2+y} + \frac{m^2}{8 + 8y + 2y^2} + \frac{m^3}{48 + 72y + 36y^2 + 6y^3}$, &c. $= x^{\frac{1}{x}}$, that is, restoring m, and involving it, &c. $1.50686 - .3086y + .18442y^2 - .1519y^3 + .10443y^4 - .0667y^5 + .0515y^6$, &c. $= x^{\frac{1}{x}}$.

And now we have four Series expressing the several Values of x^x, $\overline{x^x}|^x$, $-x^{-x}$, and $x^{\frac{1}{x}}$, the Sum of which $+ 2 + y = a$, that is, $63.6656 + 228.045y + 564.365y^2 + 1085.459y^3 + 1807.768y^4 + 2470.149y^5 + 3379.327y^6$, &c.

&c. = 200. Hence y comes out = .2681045—, therefore x = 2.2681045. Q. E. I.

PROBLEM 8.

Let $Qz^{m} + Rz^{m+n} + Sz^{m+2n} + Tz^{m+3n}$, &c. = $av^{p} + bv^{p+r} + cv^{p+2r} + dv^{p+3r}$, &c. Quere the Value of z in Terms only affected with v, and known Co-efficients, a, b, c, &c. being given Quantities; and Q. R. S. &c. either known Co-efficients, or any Powers, or Sums of Powers of the Quantity v?

Put $av^{p} + bv^{p+r} + cv^{p+2r} + dv^{p+3r}$, &c. $= Qx$; then will $x = z^{m} + \frac{Rz^{m+n}}{Q} + \frac{Sz^{m+2n}}{Q} + \frac{Tz^{m+3n}}{Q}$, &c. Assume $z = x^{\frac{1}{m}} + Bx^{\frac{n+1}{m}} + Cx^{\frac{2n+1}{m}} + Dx^{\frac{3n+1}{m}}$ &c. or $x^{\frac{1}{m}} \times \overline{1 + Bx^{\frac{n}{m}} + Cx^{\frac{2n}{m}} + Dx^{\frac{3n}{m}}, \&c.}$

And there will be $z^{m} = x \times \overline{1 + mBx^{\frac{n}{m}} + mCx^{\frac{2n}{m}} + mDx^{\frac{3n}{m}}, \&c.}$

$$m \times \frac{m-1}{2} B^{2}x^{\frac{2n}{m}} + m \times \frac{m-1}{1} BCx^{\frac{3n}{m}}, \&c.$$

$$+ m \times \frac{m-1}{2} \times \frac{m-2}{3} B^{3}x^{\frac{3n}{m}}, \&c.$$

$$z^{m+n} = x^{1+\frac{n}{m}} \times \overline{1 + \overline{m+n}Bx^{\frac{n}{m}} + \overline{m+n}Cx^{\frac{2n}{m}}, \&c.}$$

$$\frac{m+n}{1} \times \frac{m+n-1}{2} B^{2}x^{\frac{2n}{m}}, \&c.$$

&c.

z^{m+2n}

$$z^{m+2n} = x^{1+\frac{2n}{m}} \times \overline{1+\overline{m+2n}|Bx^{\frac{n}{m}}}, \ \&c.$$

&c.

$$z^{m+3n} = x^{1+\frac{3n}{m}} \times 1, \ \&c.$$

&c. = &c.

And by ſubſtituting theſe ſeveral Values in $z^{m} + Bz^{m+n}$, &c. $= x$ tranſpoſing x, and dividing the whole Equation thereby, we ſhall have

$$\left.\begin{array}{r}
mBx^{\frac{n}{m}} + mCx^{\frac{2n}{m}} + mDx^{\frac{3n}{m}}, \ \&c. \\
m \times \frac{m-1}{2} B^2 x^{\frac{2n}{m}} + m \times \frac{m-1}{1} BCx^{\frac{3n}{m}}, \&c. \\
m \times \frac{m-1}{2} \times \frac{m-2}{3} B^3 x^{\frac{3n}{m}}, \ \&c. \\
\frac{Rx^{\frac{n}{m}}}{Q} + \frac{\overline{m+n}RBx^{\frac{2n}{m}}}{Q} + \frac{\overline{m+n}RCx^{\frac{3n}{m}}}{Q}, \ \&c. \\
\frac{m+n}{1} \times \frac{m+n-1}{2} \frac{RB^2 x^{\frac{3n}{m}}}{Q}, \&c. \\
\&c. \\
\frac{Sx^{\frac{2n}{m}}}{Q} + m + \frac{2nBSx^{\frac{3n}{m}}}{Q}, \ \&c. \\
+ \frac{Tx^{\frac{3n}{m}}}{Q}, \ \&c. \\
\&c.
\end{array}\right\} = 0$$

Whence

Whence by comparing the homologous Terms, we have

$$B=-\frac{R}{mQ}\quad -C=\frac{m\times\frac{m-1}{2}B^2Q+\frac{m+n}{1}BR+S}{mQ}\quad -D=$$

$$\frac{m\times\frac{m-1}{1}BCQ+m\times\frac{m-1}{2}\times\frac{m-2}{3}B^3Q+\frac{m+n}{1}RC+\frac{m+n}{1}\times\frac{m+n-1}{2}B^2R+\frac{m+2n}{1}BS+T}{mQ},$$ &c. where the Law of Continuation is manifest.

$$\therefore z=x^{\frac{1}{m}}-\frac{Rx^{\frac{n+1}{m}}}{mQ}+\frac{\overline{m+1+2n}RR-2mSQ}{2m^2Q^2}\times x^{\frac{2n+1}{m}}$$

$$+\frac{\overline{2m^2+9mn+9nn+3m+6n+1}\times -R^3}{6m^3Q^3}+\frac{\overline{1+3n+m}\times RS-mTQ}{m^2Q^2}$$

$$\times x^{\frac{3n+1}{m}},\ \&c.$$

Which may be readily reduced into Simple Terms of v, when the Values Q, R, S, *&c.* are assigned.

COROLLARY.

Now from the above general Expression a Series more simply, expressing the Value of x, in any particular Case, may be very easily determined. For Instance,

Let $z+bz^2+cz^3+dz^4$, &c. $=x$, then by comparing this Series with $z^m+\frac{Rz^{m+n}}{Q}+\frac{Sz^{m+2n}}{Q}$, &c. $=x$, we shall have $m=1$, $n=1$, $Q=1$, $R=b$, $S=c$, &c. and by substituting these Values, in the last general Expression, it will become $z=x-bx^2+\overline{2b^2-c}\times x^3+\overline{5bc-5b^3-d}\times x^4$, &c. But if $z+bz^3+cz^5+dz^7$, &c. $=x$

$= x$; then will $m = 1$, $n = 2$, the rest as before, and consequently $z = x - bx^3 + \overline{3bb - c} \times x^5 + \overline{8cb - 12b^3 - d} \times x^7$, &c.

Likewise, if $z^2 + bz^3 + cz^4 + dz^5$, &c. $= x$; then will $z = x^{\frac{1}{2}} - \frac{bx}{2} + \frac{5bb - 4c}{8} \times x^{\frac{3}{2}} + \frac{3bc - 2b^3 - d}{2} \times x^2$, &c.

Also, if $z^2 + bz^4 + cz^6 + dz^8$, &c. $= 5x$, then will $z = \overline{5x}|^{\frac{1}{2}} - \frac{6}{2} \times \overline{5x}|^{\frac{3}{2}} + \frac{7bb - 4c}{8} \times \overline{5x}|^{\frac{5}{2}} + \frac{36bc - 33b^3 - 8d}{16} \times \overline{5x}|^{\frac{7}{2}}$, &c. the like of any other.

Q. E. I.

PROBLEM 9.

Suppose an Equation of a Curve be $a^3 \dddot{x} - a^2 \dot{x} \ddot{x} - a^2 x \dddot{x} + \dot{x} \dot{x} y^2 = 0$, and ($a$) the Radius of Evolution at the Vertex $= 100$, to find the Value of x.

By assuming $x = Ay^n$; $\dot{x} = nAy^{n-1}$, $\ddot{x} = \overline{n \times n-1}\, Ay^{n-2}$; $\dddot{x} = \overline{n \times n-1 \times n-2}\, Ay^{n-3}$, and substituting these several Values in the given Equation, it will become

$$\overline{n \times n-1 \times n-2}\, Aa^3 y^{n-3} - n^2 \times \overline{n-1}\, A^2 a^2 y^{2n-3} - \overline{n \times n-1 \times n-2}\, A^2 a^2 y^{2n-3} + nA^2 y^{2\,n-2} = 0.$$

But because the Exponent of the Power of y in the first Term, is less than in any other Term in this Equation the Value of n cannot here be had by comparing the Exponents. I therefore make the Co-efficient of that Term $= 0$, and find $n = 2$; which must be the first Exponent of the Power of y, in our required Series; and because the common Difference of the first Exponents is now found to be 2, we shall by adding that Number continually to the Value of n, have 4, 6, 8, &c. for the rest of the Exponents.

Therefore

Therefore assuming $x = Ay^2 + By^4 + Cy^6 + Dy^8$, &c. $\dot{y} = 1$, we shall have $\dot{x} = 2Ay + 4By^3 + 6Cy^5 + 8Dy^7$, &c. $\ddot{x} = 2A + 12By^2$, &c. and by writing these Values in the given Equation, making $A = \frac{1}{2}a$ (according to the *Data*) and comparing the Homologous Terms, there will come out $B = \frac{1}{4.6a^3}$, $C = \frac{1}{5.6.6a^5}$, $D = \frac{1}{7.9.10a^7}$, &c. and therefore $x = \frac{y^2}{2a} + \frac{y^4}{4.6a^3} + \frac{y^6}{5.6.6a^5} + \frac{y^8}{7.9.10a^7}$, &c. when $y = 40$, will become $8.10905 =$ Value sought.

Q. E. I.

Problem 10.

Required the Value of v, in Terms only affected with z, and known Co-efficients, in the following infinite Equation, *viz.* $-b - 2v + v^2 + \frac{v^3}{2} + \frac{v^4}{3}$, &c. $-4zv - 6z - z^2 - \frac{z^3}{2} - \frac{z^4}{0}$, &c. $= 0$. with a Method of Investigation, when in the first Term b, neither of the unknown Quantities v, z are concerned.

Let $v = A + Bz + Cz^2 + Dz^4$, &c. and substitute in the given Equation, and compare the Co-efficients, and we shall have

$$A = -\frac{1}{2}b + \frac{b^2}{8} - \frac{b^3}{8} + \frac{17}{19^2}b^4, \text{ \&c. } B = \frac{8A + 12}{3A^2 + 4A - 4},$$

$$C = \frac{2 + 8B - 2B^2}{4A - 4 + 3A^2 + 3B^2}, \quad D = \frac{1 + 8C - B^3 - 6BCA - 4BC}{3A^2 + 4A - 4}$$

Q. E. I.

N. B. If b be greater than the Co-efficient of the 2d Term (*i. e.* here -2) a different Equation must be assumed from that above; or (which may in some Cases happen as well) find the Value of z and v, and then revert the Series.

Problem

PROBLEM II.

A Gentleman meeting a Company of young Men, driving their Sheep, ask'd them, how many Sheep there was, one of whom answered; if you'll divide the Number of Sheep, amongst us equally, 'twill be double the Number we are; but if to the first Man you count one Sheep, the second, two; the third, four; the fourth, eight; and so on till the last Man, the Sum will be the Number of our Sheep. Quere the Number of each?

Let $y =$ Number Sheep, $x =$ Number of young Men, then, *per* Question, $\frac{y}{x} = 2x$, and $y = 2xx$. But in a Series of Geometrical Proportionals, wherein the first Term is $= 1$; common Ratio $= 2$; last Term $= y$; Number of Terms $= x$; $1 \times 2^{x-1} = y$, or $2xx = 2^{x-1}$, by putting for y its equal $2xx$. Now by the Nature of Logarithms, the Logarithm of the Power of any Number is equal to the Logarithm of the Root multiplied by the Index of the Power.

Suppose $6 + y = x$; and the above Equation becomes (instead of $2xx = 2^{x-1}$) $2 \times \overline{6+y}|^2 = 2^{y+5}$; or $\overline{6+y}|^2 = 2^{y+4}$.

Now to find the Hyperbolical Logarithm of $6 + y$, we must find the Fluent of $\frac{\dot{y}}{6+y} = \dot{y} \times \frac{1}{6+y}$, this will be $\dot{y} \times \frac{1}{6} - \frac{y}{36} + \frac{y^2}{216} - \frac{y^3}{1296} + \frac{y^4}{7776}$, &c. and to it add the Hyperbolical Logarithm of $6 = 1.791756 = m$. The Fluent of this is $m + \frac{y}{6} - \frac{y^2}{72} + \frac{y^3}{648} - \frac{y^4}{5184} + \frac{y^5}{38880}$. Put $n =$ Hyperbolical Logarithm

of

of 2 = .693146; then by the above Rule $2m + \frac{y}{3} - \frac{y^2}{36} + \frac{y^3}{324} - \frac{y^4}{2592} + \frac{y^5}{19440} = ny + 4n$. Let $2m - 4n = q$; $n - \frac{1}{3} = a$, and transpose the Terms. Then $ay + \frac{y^2}{36} - \frac{y^3}{324} + \frac{y^4}{2592} - \frac{y^5}{19440}$, &c. = q; or putting Letters for $\frac{1}{36}$; $\frac{1}{324}$; $\frac{1}{2592}$; $\frac{1}{19440}$, &c. $ay + by^2 - cy^3 + dy^4 - fy^5$, &c. $= q$, and by Reversion of Series, $y = \frac{q}{a} - \frac{bq^2}{a^3} + \frac{2b^2 - ac}{a^5}q^3 - \frac{a^2d - 5b^3 + 5abc}{a^7}q^4$, &c. = 2, when carried on to a sufficient Number of Places; consequently $x = 8$, the Number of young Men, and the Number of Sheep is = 128.

Q. E. I.

PROBLEM 12.

Given the Parameter of the Semi-parabola ACB (*See* Fig. 40.) = 88 *Chains, and in the same is inscribed a Right-angled Triangle. One of the angular Points of*

the other at the bound-

ht Angle is at the parabo-

Triangle is equal to half

equired the several Sides

ssa and Ordinate of the

nd solved without Flux-

= $4a = 88$; and let Fq

s, and where a Perpen-

e of the Triangle on the

Abscissa) $= x$; then $a + x = Aq$, and *per* Property of the Parabola $\overline{a + x \times 4a}|^{\frac{1}{2}} = Cq$. Now *per* E. 6. 8. as $x : \sqrt{a + x \times 4a} :: \sqrt{a + x \times 4a} : \frac{a + x \times 4a}{x} = qD$.

Consequently $FD = \frac{\overline{2a + x}^2}{x}$; whence $\frac{\frac{\overline{x + 2a}^2}{x}}{\sqrt{a + x \times 4a}}$

$\sqrt{a+x\times 4a} =$ twice Area of the Triangle, which is equal to the Semi-parabola, which is equal $a+\frac{x^2+4ax+4a^2}{x}$

$\sqrt{a+\frac{\overline{x+2a}^2}{x}, \times 4a;} \times \frac{2}{3}$, whence $x = 104.14$, &c. Consequently AD the Abscissa $= 232.729$, and DB the Semiordinate 143.109; FC $= 148.14$; and CD $=$ 139.87; required.

Q. E. I.

Problem 13.

Let DAB *be a Parabola* AC $= b$, *Parameter* $= d$, *and let* CE $= c$, *and drawing* AG *parallel to* BE, *and* EG *parallel to* AC; *then upon* G *as Centre, let a Circle be described with the Radius* GE: *It is required to find the longest Line, as* SK, *that can be drawn through both Curves, parallel to* BE; *as also (mn) the nearest Approach of the two Curves to each other.* (*See* Fig. 41.) *Fig.* 41.

In the Fig[...]
$= c$, Parame[...]
est Line, as [...]
Nature of t[...]
Nature of th[...]
Quantity; [...]

in Fluxions[...]

we have $x^3 + \frac{d}{4} x^2 = \frac{db^2}{4}$.

Now for the nearest Approach of the Curves, nm, or which is the same to find Gn. Let xGW $=$ AO; then O$n = \sqrt{dx}$, and nW $= c - \sqrt{dx}$. Consequently *per* 47. E. 1. nG $= \sqrt{dx+cc+xx-2c\sqrt{dx}}$ is a *Minimum*. In Fluxions is $d\dot{x} + 2x\dot{x} - \frac{cd\dot{x}}{\sqrt{dx}} = 0$. Reduced is

$$4x^3 + 4dx^2 + d^2x - c^2d = 0.$$

Q. E. I.

Problem

PROBLEM 14.

There is a Tree within the Arctic Circle 20.157 Yards high, that with its Shadow, on a certain Day of the Year, describes an Ellipsis, containing 9 Acres, and another Tree 40 Foot high, in the Latitude 36° 52′ N. that on the same Day, with the Shadow of its Summit, traces out such an Hyperbola, as being turned about its Axis will generate a Conoid, containing 840372 Solid Feet, betwixt its Vertex, and its Depth of 40 Foot; hence it is required to find the Sun's Declination, and the first Trees Latitude?

This Problem contains two, for the two Trees; and the latter must be solved first, to get the Sun's Declination.

I.

Let GAH be the Cone of Rays described by the Sun *Fig.* 42. in his Parallel (See *Fig.* 42.) whose Vertex is A the Top of the Tree, let the Plane of the Horizon CDB cut it in the Hyperbola BF, whose Transverse is BC, let AF be the Axis of the Cone, draw BI, CE, AD, Perpendicular to AF, and BW Perpendicular to EC, and AP Perpendicular to CB, and AP will represent the Height of the Tree; then, because its Latitude is given, there is given the Angle FAP (its Co. Lat.) which it makes with the Axis of the Cone; and its Comp. PAD. Therefore in the right-angled Triangle PAD, we have AP = 20.157 Yards, and the Angle PAD; hence is found by Trigonometry, PD, AD, and the Angle PDA. And in the Triangle BAC, here is AD bisecting BAC, and the Angle ADB given. Now to determine the Triangle BAC. It is known from the Property of Conics delivered in this Book, that IB × EC = Square of the Conjugate, whose Transverse is BC; but EW = $\frac{IB+EC}{2}$, and $\overline{BC}|^2 = \overline{BE}|^2 + \overline{CE}|^2 - 2CE \times EW = \overline{BE}|^2 - BI \times CE$; therefore $BI \times EC = \overline{BE}|^2 - \overline{BC}|^2$ = Square of the Conjugate, which call *bb*. But BE = BA+

BA + AC
= *a*, anc

proved, th

of the Hyp

this Equati

these Datas

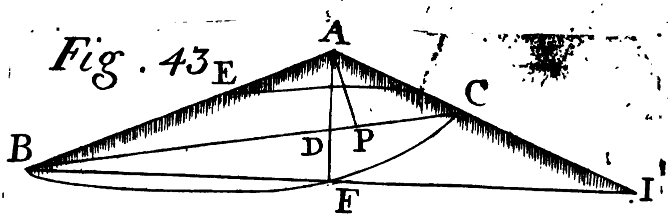
and thence

The other Problem is manag'd the same Way.

Let BC be the Transverse Axis (See *Fig*. 43.) AP the Height of the Tree, AD the Axis of the Cone. *Fig*. 43.

Then, because we have now the Sun's Declination, therefore in the Triangle BAC, we have given AP, Perpendicular to the Base, and the Angle BAC, twice the Compliment of the Sun's Declination; and consequently from this we can find another Equation from the given Area of the Ellipsis CFB; for if we put $y = BA - AC$, $BC = a$, we shall find (the same Way as before) the Conjugate $b = \sqrt{aa - yy}$; but $.7854ab$ or $.7854a \times \sqrt{aa - yy} = s$, the given Area of the Ellipsis; whence we have enough to determine all its Sides, and consequently the Angle DAP, the Compliment of the first Trees Latitude. Q. E. I.

Fig. 43



PROBLEM XV.

Use of comparing Curves *and* Fluents, *&c.*

I Shall here exhibit a new Investigation of the first Case in the 7th Proposition of Sir *Isaac Newton*'s Quadratures, applied to find the Length of an Arch of the Ellipsis, by comparing it with the correspondent Arch of the circumscribed Circle.

Put $R = e + fz^n + gz^{2n} + hz^{3n}$, &c. and let A, B, C, D, *&c.* denote the Areas of Curves, whose Ordinates are $z^{\theta-1}R^{\lambda-1}$, $z^{\theta+n-1}R^{\lambda-1}$, $z^{\theta+2n-1}R^{\lambda-1}$, $z^{\theta+3n-1}R^{\lambda-1}$, &c. respectively; then from the Principles of Quadratures, we get the following Series.

	And let us assume,
$\dot{A} = z^{\theta-1}\dot{z}R^{\lambda-1}$	$A = \frac{z^{\theta}R^{\lambda-1}}{\theta} - a$
$\dot{B} = z^{\theta+n-1}\dot{z}R^{\lambda-1}$	$B = \frac{z^{\theta+n}R^{\lambda-1}}{\theta+n} - b$
$\dot{C} = z^{\theta+2n-1}\dot{z}R^{\lambda-1}$	$C = \frac{z^{\theta+2n}R^{\lambda-1}}{\theta+2n} - c$
$\dot{D} = z^{\theta+3n-1}\dot{z}R^{\lambda-1}$	$D = \frac{z^{\theta+3n}R^{\lambda-1}}{\theta+3n} - d$

Then

Then it follows that,

$\dot{a} = \frac{\lambda-1}{\theta} z^{\theta} R^{\lambda-2} \dot{R}$

$\dot{b} = \frac{\lambda-1}{\theta+n} z^{\theta+n} R^{\lambda-2} \dot{R}$

$\dot{c} = \frac{\lambda-1}{\theta+2n} z^{\theta+2n} R^{\lambda-2} \dot{R}$

$\dot{d} = \frac{\lambda-1}{\theta+3n} z^{\theta+3n} R^{\lambda-2} \dot{R}$

And consequently

$\dot{a} z^{n} = \frac{\theta+n}{\theta} \dot{b}$

$\dot{a} z^{2n} = \frac{\theta+2n}{\theta} \dot{c}$

$\dot{a} z^{3n} = \frac{\theta+3n}{\theta} \dot{d}$

Hence, because $\dot{a} = \frac{\lambda-1}{\theta} z^{\theta} R^{\lambda-2} \dot{R}$, $a\dot{R}$ (will be) $= \frac{\lambda-1}{\theta} z^{\theta} R^{\lambda-1} \dot{R}$, in which for R in the first Part, and $\dot{R}$ in the second, put their Values $e + fz^{n} + gz^{2n} + hz^{3n}$, &c. $nfz^{n-1}\dot{z} + 2ngz^{2n-1}\dot{z} + 3nhz^{3n-1}\dot{z}$, &c. and we get $\dot{a}e + \dot{a}fz^{n} + \dot{a}gz^{2n}$, &c. $= \frac{n}{\theta} \times \overline{\lambda-1} \times \overline{fz^{\theta+n-1}\dot{z}R^{\lambda-1} + 2gz^{\theta+2n-1}\dot{z}R^{\lambda-1} + 3hz^{\theta+3n-1}\dot{z}R^{\lambda-1}}$, &c.

For $\dot{a}z^{n}$, $\dot{a}z^{2n}$, $\dot{a}z^{3n}$, &c. put their Equals from the Fourth Series, and multiply the whole Equation by θ; then we have this Equation, *viz.* $\theta e\dot{a} + \overline{\theta+n} \times f\dot{b} + \overline{\theta+2n} \times g\dot{c} + \overline{\theta+3n} \times h\dot{d}$, &c. $= n \times \overline{\lambda-1} \times \overline{fz^{\theta+n-1}\dot{z}R^{\lambda-1} + 2gz^{\theta+2n-1}\dot{z}R^{\lambda-1} + 3hz^{\theta+3n-1}\dot{z}R^{\lambda-1}}$, &c. the Fluent of which may easily be found from the first Series; then $\theta ea + \overline{\theta+n} \times fb + \overline{\theta+2n} \times gc + \overline{\theta+3n} \times hd$, &c. $= n \times \overline{\lambda-1} \times \overline{fB + 2gC + 3hD}$, &c.

For a, b, c, d, &c. put their Values from the second Series, which gives the following Equation $ez^{\theta} R^{\lambda-1} - e\theta A + fz^{\theta+n} R^{\lambda-1} - fB \times \overline{\theta+n} + gz^{\theta+2n} R^{\lambda-1} - gC \times \overline{\theta+2n} + hz^{\theta+3n} R^{\lambda-1} - hD \times \overline{\theta+3n}$, &c. $= n \times \overline{\lambda-1} \times \overline{fB+2gC+3hD}$, &c. but $ez^{\theta} R^{\lambda-1} + fz^{\theta+n} R^{\lambda-1} + gz^{\theta+2n} R^{\lambda-1}$, &c. $= z^{\theta} R^{\lambda-1} \times R = z^{\theta} R^{\lambda}$; hence $z^{\theta} R^{\lambda} - e\theta A - fB \times \overline{\theta+n} - gC \times \overline{\theta+2n} - hD \times \overline{\theta+3n}$, &c. $= n \times \overline{\lambda-1} \times \overline{fB+2gC+3hD}$, &c. or lastly, $z^{\theta} R^{\lambda} - e\theta A - fB \times \overline{\lambda n+\theta} - gC \times \overline{2\lambda n+\theta} - hD \times \overline{3\lambda n+\theta}$, &c. $= 0$; from whence the Proposition is manifest. Q. E. O.

Therefore, if the Curves are of the Binomial Kind, or $R = e+fz^{n}$; then g, h, &c. are equal to nothing, and $z^{\theta} R^{\lambda} - e\theta A - fB \times \overline{\lambda n+\theta} = 0$, or $B =$ $z^{\theta} R^{\lambda} - e\theta A$;

…as of two Bino-

…$hz^{\theta-1} R^{\lambda-1}$, and

…^{n}) then $Q = \frac{q}{p}$

…r. *Demoivre*'s six

…278, are easily de-

Fig. 44. To come to the Thing proposed; let ABG (See *Fig.* 44.) be a Semi-ellipsis, whose Transverse Axe AG is $= 2r$, Semi-conjugate Axe BN $= c$, any Ordinate to the

the Transverse ML $=y$, and its Abscifs MD $=z$; let AHG be half of the circumscrib'd Circle, which is cut by ML produced in F; call the elliptical Arch MB, E; and the circular Arch HF, C.

From the Property of the Ellipse $r^2y^2 = r^2c^2 - c^2 - c^2z^2$; ∵ $\dot{y} = \frac{c^2z\dot{z}}{r^2y}$ and $\dot{y}^2 = \frac{c^4z^2\dot{z}^2}{r^4y^2} = \frac{c^2z^2\dot{z}^2}{r^4 - r^2z^2}$, but $\dot{E} = \sqrt{\dot{z}^2 + \dot{y}^2} = \sqrt{\dot{z}^2 + \frac{c^2z^2\dot{z}^2}{r^4 - r^2z^2}}$, and therefore $\dot{E} = \dot{z}\sqrt{\frac{r^4 - z^2 \times r^2 - c^2}{r^4 - r^2z^2}}$, put $e^2 = \frac{r^2 - c^2}{r^2}$; then we have $\dot{E} = \dot{z}\sqrt{\frac{r^2 - e^2z^2}{r^2 - z^2}}$; but when $r = c$, or $e = 0$, the Ellipse becomes a Circle, therefore make $e^2 = 0$, thence $\dot{C} = r\dot{z}$.

But $\sqrt{r^2 - e^2z^2} = r - \frac{e^2z^2}{2r} - \frac{e^4z^4}{2.4r^3} - \frac{3e^6z^6}{8.6r^5} - \frac{3.5e^8z^8}{48.8r^7}$ &c. which being put for $\sqrt{r^2 - e^2z^2}$ in the Expreffion of $\dot{E}$, we have $\dot{E} = \frac{r\dot{z}}{\overline{r^2 - z^2}|^{\frac{1}{2}}} - \frac{e^2z^2\dot{z}}{2r\sqrt{r^2 - z^2}} - \frac{e^4z^4\dot{z}}{2.4r^3\sqrt{r^2 - z^2}} - \frac{3e^6z^6\dot{z}}{8.6r^5\sqrt{r^2 - z^2}} - \frac{3.5e^8z^8\dot{z}}{48.8r^7\sqrt{r^2 - z^2}}$, &c. Now let us put $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. for the Fluents of the 1st, 2d, 3d, 4th, &c. Terms of the laft Series, and (make $gz = 1$) let $\dot{\alpha}, \beta$; $\dot{\beta}, \gamma$; $\dot{\gamma}, \delta$; $\dot{\delta}, \varepsilon$, &c. be compared with the Ordinates $pz^{\theta-1}R^{\lambda-1}$, $qz^{\theta+\eta-1}R^{\lambda-1}$; From whence we get $\eta = 2$, $\lambda = \frac{1}{2}$, $e^2 = r^2$, $f = -1$, and the Values of θ are fucceffively 1. 3. 5. 7. &c. thofe of p are r, $-\frac{e^2}{2r}$, $-\frac{e^4}{2.4r^3}$, $-\frac{3e^6}{8.6r^5}$, $-\frac{3.5e^8}{48.8r^5}$ &c. and of $\frac{q}{p}$; $-\frac{e^2}{2r^2}$, $\frac{e^2}{4r^2}$, $\frac{3e^2}{6r^2}$, $\frac{5e^2}{8r^2}$, &c. which being refpectively fubftituted in the Theorem ($Q = \frac{q}{p} \times$

 pz

$\left.\frac{pz^{\theta}R^{\lambda-1-\theta}ep}{\lambda n+\theta+f}\right)$ we have the following Values of $\alpha, \beta, \gamma,$ &c.

$$\alpha = C.$$

$$\beta = -\frac{e^2}{2r^2} \times \frac{rzR^{\frac{1}{2}} - r^2\alpha}{2\times -1} = \frac{e^2 z R^{\frac{1}{2}}}{2|^2 \times r} - \frac{e^2\alpha}{2|^2}$$

$$\gamma = \frac{e^2}{4r^2} \times \overline{-\frac{e^2}{2r} \times z^3 R^{\frac{1}{2}} - 3r^2\beta} = \frac{e^4 z^3 R^{\frac{1}{2}}}{4|^2 \times 2r^3} + \frac{3e^2\alpha}{4|^2}$$

$$\delta = \frac{3e^2}{6r^2} \times \overline{-\frac{e^4 z^5 R^{\frac{1}{2}}}{2.4r^5} - 3.5e^2\gamma} = \frac{3e^6 z^5 R^{\frac{1}{2}}}{6|^2 \times 2.4r^5} + \frac{3.5e^2\gamma}{6|^2}$$

$$\varepsilon = \frac{5e^2}{8r^2} \times \frac{\overline{-\frac{3e^6z^7R^{\frac{1}{2}}}{8.6r^5} - 5.7r^2\delta}}{8\times -1} = \frac{3.5e^8z^7R^{\frac{1}{2}}}{8|^2 \times 8.6r^7} + \frac{5.7e^2\delta}{8|^2}, \text{ \&c.}$$

where the Law of Continuation is manifest; hence $E = C + \beta + \gamma + \delta + \varepsilon$, &c. Q. E. I.

COROLL

Since $R = e + fz^7 = r^2 - z^2$, th
and E is a fourth Part of the wh
Case $\alpha = C$, $\beta = \frac{-1.1e^2\alpha}{2.2}$, $\gamma = \frac{1.}{4}$
$\frac{5.7e^2\delta}{8.8}$, &c. and then $E = \alpha +$
$\frac{3.5e^2\gamma}{6.6} = \frac{5.7e^2\delta}{8.8}$, &c. whi
Purpose. Q. E. I.

PROBLEM 16.

Let A H be a Horizontal Line AZ = 100, and Z B perpendicular to AZ; and, if we suppose an infinite Number of Circles passing thro' A, and having their Centres in the Line AH, it is required to find that, along which an heavy Body, descending by the Force of Gravity, shall reach the given Perpendicular Z B in the shortest Time? (*See Fig.* 45.)

Fig. 45.

SOLUTION.

Fig. 46. Let PLqn be a Semicircle (*see Fig.* 46) whose Radius PO = 1. Call PSO, Om, x; Lq, z; and let PQE be the

he Arch required,

Then $\overline{1+x}^{\frac{1}{2}} : O^{\frac{1}{2}}$:

PQE, whose Fluxio

Question, we get $2\dot{y}$

equal to $\frac{\dot{z}}{mq^{\frac{1}{2}}}$ unive.

$\frac{\dot{x}}{1-xx^{\frac{1}{4}}} = \dot{x}$ into 1 4 4.8 4.8.12, &c.

whence $y = x + \frac{3x^3}{4.3} + \frac{3.7x^5}{4.8.5}$, &c

scribing L q by a Body already descende of its own Gravity from the Point P, 2.62221, the Time of describing P L,

$2.62221 + x + \frac{3x^3}{4.3} + \frac{3.7x^5}{4.8.5}$, &c. $= y$

Substitution above, Transposition, &c.

$\frac{3x^2}{2} + \frac{5x^3}{4} + \frac{3.7x^4}{2.8} + \frac{3.7.9x^5}{2.8.10}$, &c. = .

we have $x = .35918$.

The Radius P T =74 ferè, or =73. 69.234. Q. E. I.

PROBLEM 17.

Suppose an invariable Line $(a) = 64$ (See Fig. 47) let the Relation of the Ordinates and Abscissa be determined by this Equation, $a \times x^x = y^y$. it's required to investigate the Area ABCDA, and the Length of the Curve ABC, when the Abscissa AD=4. Fig. 47.

SOLUTION.

Make CD perpendicular to the Middle of the given Line AF (See *Fig.* 48.) and suppose mn to move uniformly from the same towards EF, put $y = mr$, or $4 + y = mn$; and $2 + x = An$: Then, by the given Equation of the Curve we have $64 \times \overline{2+x}^{2+x} = \overline{4+y}^{4+y}$; or in Logarithms $\overline{2+x}$ L : $\overline{2+x}$ + L : $64 = \overline{4+y}$ L : $4+y$, Fig. 48.

$\overline{4+y}$, that is, $x + nx + \frac{x^2}{1.2c} - \frac{x^3}{2.3c^2} + \frac{x^4}{3.4c^3}$, &c. $=$ $y + my + \frac{y^2}{2b} - \frac{y^3}{2 . 3b^2}$, &c. by putting $c = 2$, $b = 4$, $n =$ hyp. Log. of 2. $m =$ hyp. Log. 4. wherefore $y = .7066x + .0765x^2 - .0214x^3 + .0058x^4$, &c. and consequently the Fluent of $y\dot{x} = .3533x^2 + .0255x^3 -$

c. which when $x = 2$ will be
erein we write $-x$ for $+x$,
$0255x^3$, &c. $=$ Area DBgD.
$-$ BDgB $= 16 + .051x^3 +$
he required Area ABEFA.
E there is the Value of y found

$- .0961x\dot{x} - .0144x^2\dot{x}$, &c.
$048x^2 - .0048x^3$, &c. when
being doubled, and all the
Powers of x, rejected, will
. $= 4.921 =$ BDE.

Q. E. I.

LARY.

The Area of any Exponential Curve whose Nature is expressed by this exponential Equation,

$$x^x = y \text{ (making } 1 + v = x\text{) will be}$$

$$\frac{1}{0.1.2}v^2 + \frac{1}{0.1.2.3}v^3 - \frac{1}{0.1.2.3.4}v^4 + \frac{1}{0.1.2.3.4.5}v^5 - \frac{}{0.1.2.3.4.5.6}v^6, \text{ \&c.}$$

PROBLEM 18.

If, at the Time of the Earth's Arrival at her greatest Distance from the Sun, the Law of Attraction should be changed from the Square, to become as the Cube of the Distance reciprocally; but so that the Centripetal Force at that Distance should still continue the same: I desire to know

know how long, after that Change, the Earth would be before she fell into the Sun's Body, supposing the Eccentricity of her present Orbit to be 173 such Parts, whereof the Transverse Diameter is 20000?

Solution.

In order to give a compleat Solution to this Problem, it will be requisite to premise the following Lemma, because of its Use in all Questions of this Nature. The Investigation may be seen in that acute Analyst, Mr. *Simpson's* Fluxions.

Lemma.

If the Velocity of a Projectile at any Point A, at a given Distance A⊙ (a) from ⊙ the Centre of Force, be to the Velocity, that it ought to have to describe the Circle ACD at the same Distance as n to 1, and the Centripetal Force be every where as the m Power of the Distance, then the Velocity at any other Distance ⊙B (x) will be as $\left(n^2 + \frac{2}{m+1} \times a^{m+1} - \frac{2x^{m+1}}{m+1}\right)^{\frac{1}{2}}$. Let the Direction of the Projectile at the Point A (see the 49 *Fig.*) be what it will. Moreover, if the Curve AB be drawn to denote the Path of the Projectile, and the Space ⊙AB⊙ be put $=S$, the Circular Arch $AC=A$, the Sine of the Angle BA⊙ $= s$, and Radius 1; then will *Fig. 49.*

$$\frac{\frac{1}{2} na\dot{x}x}{\sqrt{\left(n^2+\frac{2}{m+1} \times x^2 - n^2s^2a^2 - \frac{2x^{m+3}}{\overline{m+1} \times a^{m+1}}\right)}} = \dot{S} \text{ and}$$

$$\frac{na^2\dot{x}x^{-1}}{\sqrt{\left(n^2+\frac{2}{m+1} \times x^2 - n^2s^2a^2 - \frac{2x^{m+3}}{\overline{m+1} \times a^{m+1}}\right)}} = \dot{A} = \frac{2a\dot{S}}{xx}$$

Now

Now to apply these Theorems to our present Purpose, let ABHRA represent the Earth's present *Elliptical* Orbit, A the Aphelion, and ABGS the Path which she will be compelled to describe, after the Law of Centripetal Force is changed. Then, because the Angle ⊙AB is a right one, and the Law of Attraction is first as the Square, and afterwards as the Cube of the Distance, inversely; by writing 1 for s, and for m, -2, and -3 successively, the Value of $\dot{S}$ will be found,

$$\frac{\frac{1}{2}nax\dot{x}}{\sqrt{2ax - n^2a^2 + \overline{nn-2} \times xx}}, \text{ and}$$

$$\frac{\frac{1}{2}nax\dot{x}}{\overline{1-nn}|^{\frac{1}{2}} \times \overline{a^2 - x^2}|^{\frac{1}{2}}} \quad \text{respectively.}$$

Now it is evident, that, whenever a right Line, drawn from a Projectile to the Centre of Force, comes to make right Angles with the Trajectory, the Value of $\dot{S}$ will then become *infinite* in respect to $\dot{x}$, or, which is the same, its Divisor will then be $= o$. Therefore, if $\overline{2ax - n^2a^2 + \overline{nn-2} + xx}|^{\frac{1}{2}}$, the Divisor of the Value of $\dot{S}$ in our first Case, be made $= o$; x will be found to have two Values, *viz.* $a =$ A⊙, and $\frac{an^2}{2-nn} =$ H⊙, the Sum of which $= \frac{2a}{2-nn} = 20000 = 2b =$ AH, must be equal to the Transverse Diameter of the Ellipsis, and half their Difference $= \frac{1-n^2 \times a}{2-nn} =$ E⊙ $= 173 = d$, its given Eccentricity; whence $n^2 = 1 - \frac{d}{b}$, and $\frac{2an}{\sqrt{2-nn}} =$ ER, the Semi-Conjugate: Therefore the Area of the Ellipsis

Ellipsis will be $= \frac{npaa}{\overline{2 - nn}|^{\frac{3}{2}}}$, p being the Area of a Circle whose Semi-diameter is Unity. In like Manner, if $\overline{a^2 - x^2}|^{\frac{1}{2}}$, the Divisor of the Value of $\dot{S}$, in our second Case, be made $= o$, x will have only one Value (a); and therefore the Trajectory can no where make right Angles with a right Line drawn from the Centre of Force, but at the Point A: Consequently, the Earth, in this Case, would be continually carried on in a Spiral ABGS, 'till at last she fell into the Sun's Body.

Now, because $\dot{A}$ is $= \frac{2\dot{S}a}{xx} = \frac{a^2 n\dot{x}}{\overline{1 - nn}|^{\frac{1}{2}} \times x\sqrt{a^2 - x^2}}$; A will be found $= \frac{\frac{1}{2}an}{\sqrt{1 - nn}} \times \text{Log.} \frac{a + \sqrt{\overline{a^2 - x^2}}}{a - \sqrt{a^2 - x^2}} = AC$, from which the Earth's new Orbit, ABGS, may be readily constructed.

Moreover, because $\dot{S}$ is $= \frac{nax\dot{x}}{2\sqrt{\overline{1 - nn}} \times \sqrt{\overline{aa - xx}}}$, S will be $= \frac{na}{2\sqrt{1 - nn}} \times \overline{a - \sqrt{\overline{aa - xx}}}$, or when $x = a = \left(\frac{naa}{2\sqrt{\overline{1 - nn}}}\right)$ the Area described about the Centre of Force, from the Alteration of the Law of Attraction, to the Time that the Earth would fall to the Sun; which is to $\frac{pa^2 n}{\overline{2 - nn}|^{\frac{3}{2}}}$, the Area of the Ellipsis before found, as $\overline{2 - nn}|^{\frac{3}{2}}$ to $2p\sqrt{\overline{1 - nn}}$: And because the Areas described are as the Times of their Description, the Times of describing those Areas will be in that same Ratio. Whence we have as $2p\sqrt{\overline{1 - nn}}$ to T, the Time of one Revolution in the present Orbit, so is $\overline{2 - nn}|^{\frac{3}{2}}$ to $\frac{\overline{2 - nn}|^{\frac{3}{2}} \times T}{2p\sqrt{1 - nn}}$ $= \frac{b + d}{2pb} \times \sqrt{1 + \frac{d}{b}} \times T$ (by substituting for n^2 its

Equal,

Equal, $1-\frac{b}{d}$, before found) = 1 Year, 87 Days, 22 Hours, nearly; and so long would it be, after the Alteration of the Law of Centripetal Force, before the Earth would fall into the Sun's Body.

SCHOLIUM.

1. Altho' the Sun in this Solution is considered as absolutely at Rest, the Error thence arising is very inconsiderable, and will not amount to one hundredth Part of a Day, by Reason of the very great Proportion which the Body of the Sun bears to that of the Earth.

2. If the Velocity at the Aphelion was to be intirely destroyed, the Earth would then fall directly along the right Line A☉, and $\frac{\overline{2-nn}|^{\frac{3}{2}} \times T}{2p\sqrt{1-nn}}$, the Time of Descent, would then become $\frac{\sqrt{2}}{p} \times \frac{1}{2\sqrt{2}} = \frac{1}{2p}$ Years (because $n=0$, and $T=\frac{1}{2\sqrt{2}}$) or 59 Days nearly.

Q. E. I.

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